A conjecture on the existence of common quadratic Lyapunov functions for positive linear systems

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Abstract

We present a conjecture concerning necessary and sufficient conditions for the existence of a common quadratic Lyapunov function (CQLF) for a switched linear system obtained by switching between two positive linear time-invariant (LTI) systems. We conjecture that these conditions are also necessary and sufficient for the exponential stability of such switched linear systems; namely, the existence of a CQLF is a non-conservative stability condition in this case. A number of new results supporting this conjecture are described.

1 Introduction

The problem of determining necessary and sufficient conditions for the existence of a common quadratic Lyapunov function (CQLF) for a set of stable linear time-invariant (LTI) systems

\[ \Sigma_{A_i} : \dot{x}(t) = A_i x(t), \quad A_i \in \mathbb{R}^{n \times n}, \quad 1 \leq i \leq k \]

plays an important role in the study of switched linear systems of the form:

\[ \dot{x}(t) = A(t)x(t), \quad A(t) \in \{A_1, ..., A_k\}. \quad (1) \]

Formally, if there is a symmetric positive definite matrix \( P \) that simultaneously satisfies the Lyapunov inequalities

\[ A_i^T P + P A_i = -Q_i < 0, \quad i \in \{1, 2, ..., k\} \quad (2) \]

then \( V(x) = x^T P x \) is a CQLF for the system (1) and the associated LTI systems \( \Sigma_{A_i} \). The existence of a CQLF is sufficient to guarantee global uniform exponential stability of (1) for arbitrary switching sequences. It is well known that requiring the existence of a CQLF for a switched linear system is, in general, a conservative stability condition [1]. However, it has recently been established that entire system classes exist for which the existence of a CQLF is not necessarily a conservative stability condition [2, 3]. In view of this observation, a problem of considerable interest and importance is to identify precisely those system classes for which the existence of a CQLF is a non-conservative stability condition. The work of this paper is primarily motivated by such considerations.

2 Notation and Preliminaries

For a matrix \( A \in \mathbb{R}^{n \times n} \), \( a_{ij} \) denotes the element in the \((i, j)\) position of \( A \), and we shall write \( A \succeq 0 \) if \( a_{ij} \geq 0 \) for \( 1 \leq i, j \leq n \). The matrix \( A \in \mathbb{R}^{n \times n} \) is said to be Hurwitz if all the eigenvalues of \( A \) have negative real parts, and for \( P \in \mathbb{R}^{n \times n} \) the notation \( P > 0 \) means that the matrix \( P \) is positive definite.

A matrix \( A \in \mathbb{R}^{n \times n} \) is a Metzler matrix if all of the off-diagonal elements of \( A \) are non-negative; that is \( a_{ij} \geq 0 \) for \( i \neq j \). The LTI system \( \Sigma_A \) is positive [4] if and only if \( A \) is a Metzler matrix. The associated class of M-matrices [5, 6] is defined to consist of matrices \( A \) with non-positive off-diagonal elements, all of whose eigenvalues lie in the open right half-plane.

A conjecture:

Let \( \Sigma_{A_i}, \ i = 1, 2 \) be a pair of stable positive LTI systems. Recent work carried out by the authors suggests that the matrix product \( A_1 A_2^T \) having no negative eigenvalues is a necessary and sufficient condition for:

(i) the existence of a CQLF for the LTI systems \( \Sigma_{A_1}, \Sigma_{A_2} \);

(ii) global exponential stability of the switched linear system (1).

3 Sufficient conditions for CQLF existence

In this section we state without proof a number of sufficient conditions for a pairs of stable positive LTI systems to possess a CQLF. Details of the proofs can be found in [7]. The result stated in the next lemma is not new [8] but is included here for the sake of comparison with the main result of this note (Theorem 3.1 below).

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Lemma 3.1 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems, with $A_1 - A_2 \geq 0$. Then $\Sigma_{A_1}$ and $\Sigma_{A_2}$ have a CQLF $V(x) = x^TPx$, with $P$ diagonal.

Theorem 3.1 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems. If both $A_1A_2^{-1}$ and $A_2^{-1}A_1$ are $M$-matrices, then $\Sigma_{A_1}$ and $\Sigma_{A_2}$ have a CQLF, $V(x) = x^TPx$, and moreover, $P$ may be taken to be a diagonal matrix.

Note that within the class of matrices with non-positive off-diagonal elements, a non-singular matrix having no eigenvalues on the negative real axis is equivalent to it being an $M$-matrix (5).

Theorem 3.2 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems. Suppose that $A_1A_2^{-1} \succeq 0$ and $A_2^{-1}A_1 \succeq 0$. Then $\Sigma_{A_1}$ and $\Sigma_{A_2}$ have a CQLF.

It was noted in [5] that if $A_1, A_2$ are both Hurwitz Metzler matrices with $A_1 \succeq 0$, then $A_1A_2^{-1}$ and $A_2^{-1}A_1$ are both $M$-matrices. Thus the class of matrices covered by Lemma 3.1 is a subclass of the class covered by Theorem 3.1. (In fact, Theorem 3.2 covers a still larger class of systems than Theorem 3.1.) The next example shows that it is a strict subclass.

Example: Consider the two Metzler matrices in $\mathbb{R}^{2 \times 2}$ given by

$$
A_1 = \begin{pmatrix}
-1.1686 & 0.5618 & 0.3837 \\
0.9512 & -1.7425 & 0.7239 \\
0.9460 & 0.4830 & -1.8474
\end{pmatrix}
$$

$$
A_2 = \begin{pmatrix}
-1.7697 & 0.3163 & 0.1496 \\
0.1599 & -0.9759 & 0.2794 \\
0.2167 & 0.1769 & -1.0543
\end{pmatrix}
$$

It is evident that $A_1 - A_2 \succeq 0$ and $A_2 - A_1 \succeq 0$ is false, so Lemma 3.1 does not apply. However, it is a simple matter to check that both $A_1A_2^{-1}$ and $A_2^{-1}A_1$ are $M$-matrices. Thus by Theorem 3.1 we can conclude that $A_1$ and $A_2$ have a CQLF $x^TPx$ with $P$ diagonal.

In [9] it is shown that LTI systems whose system matrices commute have a CQLF $x^TPx$. The next result shows that $P$ may be chosen to be diagonal if the LTI systems are positive.

Theorem 3.2 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be two positive LTI systems with $A_1A_2 = A_2A_1$. Then there is a CQLF $V(x) = x^TPx$ for $\Sigma_{A_1}, \Sigma_{A_2}$ with $P$ diagonal.

4 Conclusions

In this paper, we have proposed a conjecture concerning CQLF existence for a pair of stable positive LTI systems. It was also conjectured that for switched linear systems obtained by switching between stable positive LTI systems, the existence of a CQLF is a non-conservative stability criterion. A number of new results in this direction were presented. The authors have also gathered considerable empirical evidence supporting the conjecture.

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References


