Doing the Twist: Diagonal Meshes Are Isomorphic to Twisted Toroidal Meshes

Barak A. Pearlmutter

Abstract—We show that a $k \times n$ diagonal mesh is isomorphic to a $\frac{nk}{2} \times \frac{nk}{2}$ twisted toroidal mesh, i.e., a network similar to a standard $\frac{nk}{2} \times \frac{nk}{2}$ toroidal mesh, but with opposite handed twists of $\frac{nk}{2}$ in the two directions, which results in a loss of $(\frac{nk}{2})^2$ nodes.

Index Terms—Interconnection networks, grid networks, mesh-connected topologies, diagonal mesh, toroidal mesh.

Tang and Padubidri [1] analyze the diagonal mesh suggested by Arden, finding nonsquare diagonal meshes superior to the usual toroidal mesh in a number of respects. In Fig. 1a, the $5 \times 5$ diagonal mesh from their first figure (which is square, and thus not covered by their claims) is drawn. As shown by Fig. 1b, this network is isomorphic to a standard $5 \times 5$ toroidal mesh. A nonsquare diagonal mesh is not in general isomorphic to a standard toroidal mesh, but instead to a twisted toroidal mesh, a class of network pictured in Fig. 2.

As proven diagramatically in Fig. 3, any $k \times n$ diagonal mesh ($n$ and $k$ are necessarily odd, and without loss of generality $k \leq n$) is isomorphic to a $\frac{nk}{2} \times \frac{nk}{2} - \frac{nk}{2} \times \frac{nk}{2}$ twisted toroidal mesh. This twisted toroidal mesh is like a standard $\frac{nk}{2} \times \frac{nk}{2}$ toroidal mesh, except that the edges are joined with twists of opposite handedness of $\frac{nk}{2}$ in the two directions, and there is a consequent loss of an $\frac{nk}{2} \times \frac{nk}{2}$ corner, as shown in Fig. 2.

A convenient notation for a $k \times n$ toroidal mesh with twists of $a$ and $b$ in the two directions is $k \times n \pm a \times b$, with $+$ if the twists have the same handedness and $-$ if they have opposite handedness. This notation serves a dual purpose, as such networks have $kn = ab$ nodes.

This simplifies the analysis of diagonal meshes. For instance, for a large network, holding the number of nodes in a $k \times n \pm a \times b$ twisted toroidal mesh fixed while allowing $k$, $n$, $a$, and $b$ to vary, it is elementary to see that the bisection width and diameter reach extremes at the discontinuities of the domain, namely configurations of the form $n \times n \pm \frac{a}{2} \times \frac{b}{2}$. The extreme which optimizes performance is with an $n \times n - \frac{a}{2} \times \frac{b}{2}$, and is isomorphic to a $k \times 3k$ diagonal mesh, and therefore to a $2k \times 2k - k \times k = k \times k - 0 \times 2k = k \times 0 + 0 \times 2k$ twisted toroidal mesh.

We call a network singularly transversible when, by moving repeatedly in one direction, all nodes will be visited. This property can be useful for testing, power distribution, diagnosis, and initialization. A $k \times n \pm a \times b$ twisted toroidal mesh is singularly transversible exactly when $\gcd(k, a) = \gcd(n, b) = 1$. This implies that a $k \times n$ diagonal mesh is singularly transversible when $k$ and $n$ are relatively prime.

Consider a $k \times n - a \times b$ twisted toroidal mesh as an Abelian group. This group can be generated by the two elements $N$ and $E$. Two identities suffice to characterize its properties: $F' = N'$ and $F'' = E'$, where

$$k' = \gcd(k, a) \quad n' = \frac{kn - ab}{k}$$

and to preserve the group identities, it is necessary that $b' = b' = b'(mod n')$ and $b' = b'(mod n')$. Using the Chinese remainder theorem, we can find integers $x$ and $y$ such that $x \frac{a}{k} + y \frac{b}{n} = 1$, so $b' = xb + y = (mod n')$

This gives a simple algorithm for testing twisted toroidal mesh isomorphism.

A twisted toroidal topology was used as the routing network of the FAIM-1 parallel computer [2]. Fig. 4 shows the 19-element E3 hex-mesh toroidal network they built. If all the links in one of the three directions are removed, what remains is a $5 \times 5 - 2 \times 3$ twisted toroidal mesh.

Diagramatic Proof of Isomorphism

We begin with an arbitrary diagonal mesh network. For clarity a $5 \times 7$ net is shown. Identifed nodes are indicated by shading.

Fig. 1. (a) $5 \times 5$ diagonal mesh. The edge behavior is shown by ghost units and corresponding regions. (b) $5 \times 5$ toroidal mesh. As demonstrated by the node labels, these two networks are isomorphic.

Fig. 2. Removing an $a \times b$ rectangular area from the corner of a $5 \times 5$ rectangular mesh before joining the edges to form a torus allows opposite handed twists of $a$ and $b$ nodes to be made in the two directions, resulting in an $a \times b$ twisted toroidal mesh, which has $kn - ab$ nodes. Edge identifications are shown on the left using shaded regions, while the process of rolling the sheet up is shown on the right.

$$N' = \frac{kn - ab}{k}$$

1. Isomorphic in the sense that there exists a bijective mapping of nodes to nodes, edges to edges, and directions to directions that preserves all mathematical properties.

* The author is with the Department of Cognitive Science, University of California, San Diego, UCSD Mail Stop 0515, 9500 Gilman Drive, La Jolla, CA 92039-0515. E-mail: barak@pearlmutter@cognitiv.cs.ucsd.edu.

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Even and odd parity nodes are segregated and joined at an identified edge.

The surface is redrawn and symbols used to indicate identified edges:

The surface is cut into four regions:

The regions are rearranged, revealing the isomorphism to a twisted toroidal mesh:

Fig. 3. This diagram sketches a simple proof that a $k \times n$ diagonal mesh network is isomorphic to a $\frac{n-k}{2} \times \frac{n-k}{2} - \frac{n-k}{2} \times \frac{n-k}{2}$ twisted toroidal mesh.

Fig. 4. Hex-mesh regular hexagonal arrays can be rolled into twisted toruses, as in this 19-element E3 network [2].

REFERENCES
