AN UNUSUAL CONSTRUCTION
OF THE FOURIER TRANSFORM

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Introduction
The usual construction of the Fourier transform involves working on $L^1(\mathbb{R})$ (eg. [1]) or the Schwartz Class $S$ of rapidly decreasing $C^\infty$ functions (eg. [2]). The Fourier transform is then extended onto $L^2(\mathbb{R})$ by taking limits as both $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $S$ are dense in $L^2(\mathbb{R})$.

I’d like to present an unusual construction of the Fourier transform in which we use its translation and dilation properties:

- if $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega} \hat{f}(\omega)$,
- if $g(x) = f(\lambda x)$ then $\hat{g}(\omega) = 1/|\lambda| \hat{f}(\omega/\lambda)$.

This construction is in the spirit of the “multiresolution analysis” structure [3] which is used to build discrete wavelet bases [4]. However, if you don’t know anything about this structure the construction is still surprisingly straightforward.

The Definition
I’ll be using both $\hat{f}$ and $\mathcal{F}f$ to denote the Fourier transform of $f$.

For translation and dilation I’ll write:

- $(T_\alpha f)(x) := f(x + \alpha)$ for any $\alpha \in \mathbb{R}$,
- $(D\lambda f)(x) := f(\lambda x)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$,
- $(R\alpha f)(x) := e^{i\alpha x} f(x)$ for any $\alpha \in \mathbb{R}$.

This allows me to write the translation and dilation properties as

$$\mathcal{F}T_\alpha f = R_\alpha \mathcal{F} f$$

and
\[ \mathcal{F}D_\lambda f = \frac{1}{|\lambda|} D_\lambda^\ast \mathcal{F} f. \]

Now to define \( \mathcal{F} \), we do the following:

1. for the characteristic function of \([0, 1)\) we define

   \[ \mathcal{F}(\chi_{[0, 1)}) = \frac{1 - e^{-i\omega}}{i\omega}, \]

2. we extend \( \mathcal{F} \) to any \( \chi_{(n, n+1)} \) by using the translation rule

   \[ \mathcal{F}T_n f = R_n \mathcal{F} f \]

   for all \( n \in \mathbb{Z} \),

3. we further extend \( \mathcal{F} \) to functions of the form \( \chi_{[\lambda^{-1} n, \lambda^{-1} (n+1))} \) by using the dilation rule

   \[ \mathcal{F}D_\lambda f = \frac{1}{|\lambda|} D_\lambda^\ast \mathcal{F} f \]

   for all \( \lambda \in 2\mathbb{Z} \).

4. finally we extend \( \mathcal{F} \) to the linear span of these by assuming that \( \mathcal{F} \) is linear.

If this process works, we have defined \( \mathcal{F} \) on \( D \), the set of dyadic step functions. These are just the simple functions whose jumps occur at \( n/2^m \) where \( n, m \in \mathbb{Z} \). It is easy to construct a well defined function which has these properties. In fact the definition spells out a formula:

\[ f(x) = \sum_{r=-R}^{R} a_r \chi_{[0, 1)}(2^j x - r) \Rightarrow \]

\[ \mathcal{F}f(\omega) = \frac{1 - e^{-i\omega}}{i\omega} \sum_{r=-R}^{R} a_r e^{-i\omega r/2^j}. \]

Our aim was to produce \( \mathcal{F} \) on \( L^2(\mathbb{R}) \). Given that the set of simple functions is dense in \( L^2(\mathbb{R}) \) it is clear that \( D \) is also dense in
$L^2(\mathbb{R})$. So, if we can show that this function $\mathcal{F}$ we have defined is continuous in the $L^2(\mathbb{R})$ norm then we can extend $\mathcal{F}$ to all of $L^2(\mathbb{R})$.

This turns out to be surprisingly straight forward. Taking

$$f(x) = \sum_{r=-R}^{R} a_r \chi_{(0,1)}(2^j x - r)$$

we see that $\|f\|^2_2 = \sum_{r=-R}^{R} |a_r|^2 / 2^j$.

Now we have to find $\|\mathcal{F}f\|^2_2$. Using our formula above and the definition of the norm:

$$\|\mathcal{F}f\|^2_2 =$$

$$\int_{-\infty}^{\infty} \left| \frac{1 - e^{-i \frac{2\pi}{2^j} r}}{i \omega} \right|^2 \left( \sum_{k=-N}^{N} a_k e^{-i \frac{2\pi}{2^j} k} \right) \left( \sum_{l=-N}^{N} \overline{a_l} e^{i \frac{2\pi}{2^j} l} \right) d\omega =$$

$$\int_{-\infty}^{\infty} \frac{2(1 - \cos \frac{2\pi}{2^j} r)}{\omega^2} \left[ \left( \sum_{k=-N}^{N} |a_k|^2 \right) + \sum_{k \neq l} a_k \overline{a_l} e^{-i (k-l) \frac{2\pi}{2^j}} \right] d\omega.$$

So we need to evaluate:

$$\int_{-\infty}^{\infty} \frac{2(1 - \cos \omega)}{\omega^2} e^{i r \omega} d\omega,$$

for $r \in \mathbb{Z}$. This is an easy piece of contour integration, giving $2\pi$ if $r = 0$ and zero otherwise. Filling this in we see:

$$\|\mathcal{F}f\|^2_2 = \sum_{r=-R}^{R} |a_r|^2 / 2^j 2\pi = 2\pi \|f\|^2_2.$$

So, not only is $\mathcal{F}$ continuous but it just scales the norm. This means that we may extend $\mathcal{F}$ to a continuous map from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ which preserves the inner product:

$$(f,g) = 2\pi(\mathcal{F}f, \mathcal{F}g).$$
What now?

Note that we could show that $\mathcal{F}$ as defined on $D$ also extends to a continuous map $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ by examining the $L^1(\mathbb{R})$ norm of $f$ and the $L^\infty(\mathbb{R})$ norm of $\mathcal{F}f$. This might motivate us to try to get the usual integral formula for the Fourier transform back again.

This can be done for $f \in D$ by first considering $f$ as a function with steps of width $2^{-j}$, and then splitting each step in half to get the same function written in terms of steps of width $2^{-(j+1)}$. This turns our formula for $\mathcal{F}$ into a Riemann sum for the integral:

$$(\mathcal{F}f)(\omega) = \int f(x)e^{-i\omega x} \, dx \quad f \in D.$$ 

This can naturally be extended to suitable sets larger than $D$.

By looking at the dilation and translation relations carefully (or by using the integral formula) we get extended translation and dilation, this time for all $f \in L^2(\mathbb{R})$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$:

- $\mathcal{F}T_{\alpha}f = R_{-\alpha} \mathcal{F}f$,
- $\mathcal{F}D_{\lambda}f = \frac{1}{|\lambda|} D_{\frac{\lambda}{|\lambda|}} \mathcal{F}f$,
- $\mathcal{F}R_{\alpha}f = T_{-\alpha} \mathcal{F}f$.

This provides us with a neat way to show that $\mathcal{F}$ is invertible on $L^2(\mathbb{R})$. Suppose we defined $\mathcal{G}$ with the translation and dilation properties we expect of $\mathcal{F}^{-1}$. Then by proceeding as we did for $\mathcal{F}$, we arrive at an integral formula and the following properties for $\mathcal{G}$:

- $\mathcal{G}T_{\alpha}f = R_{-\alpha} \mathcal{G}f$,
- $\mathcal{G}D_{\lambda}f = \frac{1}{|\lambda|} D_{\frac{\lambda}{|\lambda|}} \mathcal{G}f$,
- $\mathcal{G}R_{\alpha}f = T_{-\alpha} \mathcal{G}f$.

We examine $I = \mathcal{F} \mathcal{G}$ and how it interacts with $T_n$, $D_\lambda$ and $\chi_{[0,1]}$. Using the algebraic properties of $\mathcal{F}$ and $\mathcal{G}$:

$$IT_n f = \mathcal{F} \mathcal{G} T_n f = \mathcal{F} \mathcal{R}_{\alpha} \mathcal{G} f = T_n \mathcal{F} \mathcal{G} f = T_n I f$$

$$ID_\lambda f = \mathcal{F} \mathcal{G} D_\lambda f = \frac{1}{|\lambda|} \mathcal{F} \mathcal{D}_{\frac{\lambda}{|\lambda|}} \mathcal{G} f = |\lambda| \mathcal{D}_{\lambda} \mathcal{F} \mathcal{G} f = D_\lambda I f.$$
Thus $I$ commutes with $T_n$ and $D_\lambda$, so if we can determine the image of $\chi_{(0,1)}$ we can determine the image of $D$. Using the integral formula and a little contour integration we see:

$$(I\chi_{(0,1)})(x) = (\mathcal{F}\mathcal{G}\chi_{(0,1)})(x)$$

$$= \mathcal{F}\left(\frac{e^{i\omega} - 1}{i\omega}\right)$$

$$= \int \frac{e^{i\omega} - 1}{i\omega}e^{-i\omega x} \, d\omega = 2\pi \chi_{(0,1)}(x)$$

for almost every $x$. So $I$ acts on $D$ by multiplying by $2\pi$. Using the fact that $I$ is continuous we see that $I$ acts on all of $L^2(\mathbb{R})$ in this way, and so $(2\pi)^{-1}I$ is a right inverse for $\mathcal{F}$. Naturally a similar argument shows that it is also a left inverse.

**To finish up**

This is a curious construction of the Fourier transform. It is even quite easy to extend it to $L^2(\mathbb{R}^n)$. One interesting point I didn’t touch on is that we may change the first rule with which we defined $\mathcal{F}$ from:

- for the characteristic function of $[0, 1)$ we define

$$\mathcal{F}(\chi_{(0,1)}) = \frac{1 - e^{-i\omega}}{i\omega},$$


to the seemingly weaker:

- $\mathcal{F}(\chi_{(0,1)})$ is continuous at zero and has value 1 at zero.

This is because $\chi_{(0,1)}$ satisfies the dilation equation:

$$\chi_{(0,1)}(x) = \chi_{(0,1)}(2x) + \chi_{(0,1)}(2x - 1),$$

but that is another story, [5].

**References**


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