SYLOW INTERSECTIONS, DOUBLE COSETS, AND 2-BLOCKS

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1. NOTATION AND STATEMENT OF RESULTS

Throughout $G$ will be a finite group and $F$ will be a finite field of characteristic $p > 0$, although we are mainly interested in the case $p = 2$. For convenience we assume that $F$ is a splitting field for all subgroups of $G$. We let $\mathbb{Z}_p$ denote the localization of the integers $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. If $x \in \mathbb{Z}_p$, then $x^*$ will denote its image modulo the unique maximal ideal of $\mathbb{Z}_p$. We regard $x^*$ as lying in the prime field $\text{GF}(p)$ of $F$.

The elements of $G$ may be identified with the members of a distinguished basis of the group algebra $FG$. Thus each $x \in FG$ is of the form $x = \sum_{g \in G} \beta(x, g) g$, where $\beta(x, g) \in F$, for $g \in G$. We define the element $x^o$ of $FG$ as $x^o := \sum_{g \in G} \beta(x, g^{-1}) g$. The map $x \to x^o$ is an anti-isomorphism of $FG$, and its restriction to the centre $Z$ of $FG$ is an involutary isomorphism. We use $\mathcal{K}^+$ to denote the sum of the elements in a $G$-conjugacy class $\mathcal{K}$ in $FG$.

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The set of all such class sums forms an $F$-basis for $Z$. If $x \in Z$ and $g \in \mathcal{K}$, we will use $\beta(x, \mathcal{K}^+)$ in place of $\beta(x, g)$.

By a $p$-block $B$ of $G$ we mean a direct $F$-algebra summands of $FG$. Associated with $B$ there is a primitive idempotent $e \in Z$, and an $F$-epimorphism $\omega : Z \to F$. We indicate these associations by $B \leftrightarrow e \leftrightarrow \omega$, and call $e$ the block idempotent, and $\omega$ the central character, of $B$. Set $B^o := \{ x \in FG \mid x^o \in B \}$ and $\omega^o(x) := \omega(x^o)$. Then $B^o \leftrightarrow e^o \leftrightarrow \omega^o$ is a $p$-block of $G$. We say that $B$ is a real block of $G$ if $B = B^o$.

R. Brauer showed how to associate with $B$ a $G$-conjugation family of $p$-groups, which he called the defect groups of $B$. Let $D$ be a defect group of $B$. Then $D$ is not arbitrarily embedded in $G$. For instance Brauer proved that $D$ is the largest normal $p$-subgroup of its normalizer $N(D)$. J. A. Green [5] showed that there exists $g \in G$ and a Sylow $p$-subgroup $S$ of $G$, such that $S \cap S^g = D$, and M. F. O’Reilly [9] showed that $g$ could be chosen to be $p$-regular with defect group $D$. Here a defect group of $g$ means a Sylow $p$-subgroup of the centralizer $C(g)$ of $g$ in $G$.

Let $e_D$ denote the sum of the block idempotents associated with the $p$-blocks of $G$ which have defect group $D$, and let $Syl$ denote the collection of Sylow $p$-subgroups of $G$. We prove the following partial converse to these results:

**Theorem 2.9.** Let $p = 2$ and let $g \in G$ be 2-regular with defect group $D$. Then $\beta(e_D, g) = \left| \{ P \in Syl \mid P \cap P^g = D, PgP = Pg^{-1}P \} \right|_F$. 

Now $G$ has a 2-block with defect group $D$ if and only if $\beta(e_D, g) \neq 0$ for some 2-regular element $g$ with defect group $D$. So 2.9 furnishes a necessary and sufficient condition for $G$ to have a 2-block with defect group $D$. 

If $g \in G$, set $\mathbf{C}^*(g) := \{x \in G \mid g^x \in \{g, g^{-1}\}\}$. We call the Sylow 2-subgroups of $\mathbf{C}^*(g)$ the extended defect groups of $g$. Let $K$ be the conjugacy class of $G$. The extended defect groups of the elements of $K$ form a single $G$-orbit, which we call the extended defect groups of $K$. If $E$ is an extended defect group of $g$, then $D := \mathbf{C}_E(g)$ is a group of $g$ which is contained in $E$, and $|E : D| \leq 2$. We call $(D, E)$ a defect pair for $g$. The defect pairs of the elements of $K$ form a single $G$-orbit. We call $g$ a real element if it is $G$-conjugate to $g^{-1}$. Theorem 2.9 can be refined for real elements as follows:

**Theorem 3.1.** Let $p = 2$ and let $g \in G$ be real and 2-regular with defect pair $(D, E)$. Then $\beta(\epsilon_D, g) = |\{P \in \mathcal{S}_g \mid P \cap P^g = D, E \leq P\}|1_F$.

We use this theorem to give an alternative proof of Theorem 4.8 of [4] and also to provide a self-contained treatment of some results of M. Herzog.

Let $K$ be a conjugacy class of $G$. We call $K$ a real class if it coincides with its inverse class $K^o := \{g \in G \mid g^{-1} \in K\}$. We call a real class $K$ properly real if $g^2 \neq 1$ for $g \in K$.

Suppose that $K$ and $L$ are conjugacy classes of $G$. We write

$$K \leq L,$$

if each defect group of $K$ is contained in some defect group of $L$. Suppose in addition that $K$ is properly real and that $L$ is real. We write

$$K \preceq L,$$

if for each defect pair $(D, E)$ of $K$, there exists $l \in L$ such that $D \leq \mathbf{C}(l)$ and $E \leq \mathbf{C}^*(l)$, but $E \not\leq \mathbf{C}(l)$ if $l^2 \neq 1$.

Let $B \leftrightarrow e \leftrightarrow \omega$ be a 2-block of $G$. It is well-known that there exists a 2-regular class $L$ of $G$ such that $\beta(e, L^+) \neq 0$ and $\omega(L^+) \neq 0$. Any such $L$ is
called a \textit{defect class} for $B$. The min-max theorem \cite[15.31]{7} states that
\begin{equation}
\omega(\mathcal{K}^+) \neq 0 \implies \mathcal{L} \leq \mathcal{K}, \quad \text{for each class } \mathcal{K} \text{ of } G.
\end{equation}
\begin{equation}
\beta(e, \mathcal{K}^+) \neq 0 \implies \mathcal{K} \preceq \mathcal{L},
\end{equation}

Suppose that $B$ is a real 2-block of $G$. R. Gow showed in \cite{3} that $B$ has a defect class $\mathcal{K}$ which is real. Let $(D, E)$ be a defect pair for $\mathcal{K}$. Gow proved that the extended defect groups of the real defect classes of $B$ are $G$-conjugate to $E$. For this reason he referred to the $G$-conjugates of $E$ as the extended defect groups of $B$. We call $(D, E)$ a \textit{defect pair} for $B$. Theorem 2.1 of \cite{3} can be extended in the following way:

\textbf{Theorem 4.3 (Min-Max for Real 2-Blocks).} Let $B \leftrightarrow e \leftrightarrow \omega$ be a real non-principal 2-block of $G$ and let $\mathcal{L}$ be a real defect class of $B$. Then
\begin{equation}
\omega(\mathcal{K}^+) \neq 0 \implies \mathcal{L} \preceq \mathcal{K}, \quad \text{for each real class } \mathcal{K} \text{ of } G.
\end{equation}
\begin{equation}
\beta(e, \mathcal{K}^+) \neq 0 \implies \mathcal{K} \preceq \mathcal{L},
\end{equation}

Throughout the paper $S$ will be a fixed Sylow 2-subgroup of $G$. Let $D \leq E$ be subgroups of $S$ with $|E : D| = 2$. Let $S \setminus G/S$ denote a set of representatives for the double cosets of $S$ in $G$. If $x, y \in G$ lie in the same $(S, S)$-double coset, then the groups $S \cap S^x$ and $S \cap S^y$ are $S$-conjugate. We say that $SgS$ is a \textit{self-dual double coset} if $SgS = Sg^{-1}S$. Lemma 5.1 furnishes a 2-subgroup $(S \cap S^y)^*$ of $G$ which contains $S \cap S^y$ as a subgroup of index 2, whenever $SgS$ is self-dual and distinct from $S$. Moreover, if $x, y \in SgS$, then $(S \cap S^x)^*$ and $(S \cap S^y)^*$ are conjugate in $S$.

If $H$ and $K$ are subgroups of $G$, we write $H =_G K$ if some $G$-conjugate of $H$ equals $K$, and we write $H \leq_G K$ if some $G$-conjugate of $H$ is contained in $K$. We let $\sum_x^D$ denote a sum which ranges over those elements $x$ of $S \setminus G/S$
for which $S \cap S^x = G D$, and let $\sum_{x \equiv x-1}^D$ denote the restriction of this sum to the self-dual double cosets.

Suppose that $\{K_1, \ldots, K_r\}$ is a complete list of the real 2-regular classes of $G$ which have defect pair $(D, E)$. We call a self-dual double coset $SgS$ a 

$(D, E)$-double coset if there exists $x \in Sg$ which simultaneously satisfies:

1. $x \in K_1 \cup \cdots \cup K_r$;
2. $(S \cap S^x, (S \cap S^x)^*)$ is a defect pair for $x$.

Let $x_1, \ldots, x_w$ be a (possibly empty) set of representatives for the $(D, E)$-double cosets of $S$. Suppose that $w \neq 0$. We define an $v \times w$ integer matrix $N$ by setting the $i, j$-th entry of $N$ to be the number $N_{ij}$ of $y_k$ in $K_i \cap Sx_j$ such that $(S \cap S^x_i, (S \cap S^x_i)^*)$ is a defect pair for $y_k$. When $p = 2$, Theorem A of [11] can be refined as follows:

**Theorem 5.2.** The number of real 2-blocks of $G$ which have defect pair $(D, E)$ is zero, if $w = 0$, and is the 2-rank of the matrix $N \cdot N^T$, if $w \neq 0$.

2. SYLOW INTERSECTIONS AND 2-BLOCKS

Our starting point is Proposition 3.1 of [4]:

**Lemma 2.1.** Let $B \leftrightarrow e \leftrightarrow \omega$ be a $p$-block of $G$ which has defect group $D$ and let $K$ be a $p$-regular class of $G$ which has defect group $D$. Then

$$
\beta(e, K^+) = \left( \frac{\dim(B)}{|G|} \right)^* \omega(K^o+).
$$

Let $JZ$ denote the Jacobson radical of $Z$ and let $EZ$ denote the $F$-span of the idempotents in $Z$. Then $JZ$ is an ideal of $Z$, $EZ$ is a direct sum of copies of $F$ (as an $F$-algebra) and $Z = JZ \bigoplus EZ$ as $F$-algebras. The Robinson map is the natural $F$-algebra projection $e : Z \to EZ$ with respect to this
decomposition. Let \( z \in Z \) and let \( n \) be a positive integer such that \( g^{p^n} = 1_G \), for each \( p \)-element \( g \) of \( G \), and \( \lambda^{p^n} = \lambda \), for each \( \lambda \in F \). Then \( \epsilon(z) = z^{p^n} \). We also have

\[
\epsilon(z) = \sum_B \omega(z)e,
\]

where \( B \leftrightarrow e \leftrightarrow \omega \) ranges over the \( p \)-blocks of \( G \). See [12] for further details.

For the rest of the paper we take \( p = 2 \) and \( \text{Char}(F) = 2 \). Recall that \( e_D \) denotes the sum of the block idempotents in \( Z \) which have defect group \( D \). We combine Lemma 2.1 and (2.2) as follows:

**Corollary 2.3.** Let \( \mathcal{K} \) be a 2-regular class of \( G \) which has defect group \( D \). Then \( \beta(e_D, \mathcal{K}^+) = \beta(e(\mathcal{K}^+), \mathcal{K}^+) \).

**Proof.** It follows from (1.2) and (2.2) that

\[
\beta(e(\mathcal{K}^+), \mathcal{K}^+) = \sum_B \omega(\mathcal{K}^+)^2 \beta(e, \mathcal{K}^+),
\]

where \( B \leftrightarrow e \leftrightarrow \omega \) ranges over the 2-blocks of \( G \) which have defect group \( D \). Also \( (\dim(B)/|G||\mathcal{K}|)^* = 1_F \), for each such \( B \). Thus

\[
\beta(e(\mathcal{K}^+), \mathcal{K}^+) = \sum_B \beta(e, \mathcal{K}^+)^2, \quad \text{using Lemma 2.1}
\]

\[
= (\sum_B \beta(e, \mathcal{K}^+))^2, \quad \text{as } F \text{ has characteristic 2}
\]

\[
= \beta(e_D, \mathcal{K}^+)^2
\]

\[
= \beta(e_D, \mathcal{K}^+), \quad \text{as } \beta(e_D, \mathcal{K}^+) \in \text{GF}(2).
\]

\[ \square \]

If \( \mathcal{K} \) and \( \mathcal{L} \) are 2-regular classes which have defect group \( D \) then

\[
\beta(e(\mathcal{K}^+), \mathcal{L}^+) = \sum_x |\mathcal{K} \cap Sx| |\mathcal{L} \cap Sx| 1_F,
\]

where \( x \) ranges over the 2-blocks of \( G \) which have defect group \( D \).
using 1.3.3 and 1.3.4 of [12]. This allows us to prove:

**Proposition 2.5.** Let $\mathcal{K}$ be a 2-regular class with defect group $D$. Then

$$\beta(e_D, \mathcal{K}^+) = \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx| 1_F.$$  

**Proof.** By Corollary 2.3 and (2.4) we have

$$\beta(e_D, \mathcal{K}^+) = \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx| |\mathcal{K} \cap Sx| 1_F. \tag{2.6}$$

Let $x \in G$. The map $sx \leftrightarrow (sx)^{-s}$, for $s \in S$, establishes a bijection between the sets $\mathcal{K} \cap Sx$ and $\mathcal{K}^o \cap Sx^{-1}$. So

$$|\mathcal{K} \cap Sx| = |\mathcal{K}^o \cap Sx^{-1}|. \tag{2.7}$$

Suppose $SxS \neq Sx^{-1}S$. Then the contribution of these cosets to (2.6) is

$$|\mathcal{K}^o \cap Sx| |\mathcal{K} \cap Sx| 1_F + |\mathcal{K}^o \cap Sx^{-1}| |\mathcal{K} \cap Sx^{-1}| 1_F = 2 |\mathcal{K}^o \cap Sx| |\mathcal{K} \cap Sx| 1_F = 0_F.$$  

It follows that

$$\beta(e_D, \mathcal{K}^+) = \sum_{x \in x^{-1}} |\mathcal{K}^o \cap Sx| |\mathcal{K} \cap Sx| 1_F. \tag{2.8}$$

Suppose that $SxS = Sx^{-1}S$. Then

$$|\mathcal{K}^o \cap Sx| = |\mathcal{K}^o \cap Sx^{-1}|,$$  

as $Sx^{-1}$ and $Sx$ are $S$-conjugate

$$= |\mathcal{K} \cap Sx|, \quad \text{by (2.7).}$$

We conclude from (2.8) and the fact that the prime field of $F$ is GF(2) that

$$\beta(e_D, \mathcal{K}^+) = \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx|^2 1_F = \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx| 1_F.$$  

$\square$
Proof of Theorem 2.9. Recall that \( g \) is a 2-regular element of \( G \) with defect group \( D \). Let \( \mathcal{K} \) be the class of \( G \) which contains \( g \). We shall compute \(|\{(k, P) \in \mathcal{K} \times \mathcal{Syl} | PkP = Pk^{-1}P, P \cap P^g = D\}|\) in two different ways. On the one hand it equals \(|\mathcal{K}| \mu(g)|\), where

\[
\mu(g) := \{ P \in \mathcal{Syl} | PgP = P g^{-1}P, P \cap P^g = D\}.
\]

On the other hand it equals \(|\mathcal{Syl}| \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx|\). The double coset \( SxS \) is a union of \(|S : S \cap S^x|\) right cosets of \( S \), and each of these is \( S \)-conjugate to \( Sx \).

It follows that \(|\mathcal{K} \cap SxS| = |S : S \cap S^x| |\mathcal{K} \cap Sx|\). Also \(|S : S \cap S^x| = |S : D|\), whenever \( S \cap S^x \) is \( G \)-conjugate to \( D \). But \(|\mathcal{Syl}| \) is odd, by Sylow’s Theorem. Thus

\[
|\mu(g)|_F = \frac{|S : D|}{|\mathcal{K}|} \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx|_F \\
= \sum_{x \in x^{-1}} |\mathcal{K} \cap Sx|_F, \quad \text{as } \mathcal{K} \text{ has defect group } D \\
= \beta(e_D, g), \quad \text{by Proposition 2.5.}
\]

We claim that \( D \) acts by conjugation on \( \mu(g) \). For, suppose that \( P \in \mu(g) \) and \( d \in D \). Then \( dg = gd \). So \( P^d \cap P^{g} = (P \cap P^{g})^d = D \), and \( P^d g P^d = (P g P)^d = (P g^{-1} P)^d = P^d g^{-1} P^d \). Thus \( P^d \in \mu(g) \), which proves our claim.

Each \( D \)-orbit in \( \mu(g) \) has 2-power order, and \( P \) is stabilized by \( D \) if and only if \( D \leq P \). But \( D \leq P \) implies that \( D \leq P \cap P^g \). Since \( P \cap P^g = D \), it follows that \( P \) is stabilized by \( D \) if and only if \( P \cap P^g = D \). We conclude that

\[
|\mu(g)| \equiv |\{ P \in \mathcal{Syl} | P \cap P^g = D, PgP = Pg^{-1}P\}| \pmod{2},
\]

from which the theorem follows.
Theorem 2.9 has no obvious analogue for odd primes. For instance, if $P$ is a Sylow 3-subgroup of $\text{PSL}_3(2)$ and $g$ is an element of order 4, the set 
\[ \{P \in \text{Syl} \mid P \cap P^g = \{1\}, PgP = Pg^{-1}P\} \] 
has cardinality 4. However, $g$ has 3-defect zero and appears with zero multiplicity in the sum of the 3-block idempotents of defect zero.

We indicate how our methods may be used to sharpen Corollary 2 of [11]:

**Theorem 2.10.** Let $g$ be a 2-regular element of $G$ which has defect group $D$. Suppose that $P \cap P^g = D$, for each Sylow 2-subgroup $P$ of $G$ which contains $D$. Then $g$ lies in a defect class of some real 2-block of $G$. In particular, $G$ has a real 2-block with defect group $D$.

**Proof.** Let $r_D$ denote the sum of the real 2-block idempotents of $G$ which have defect group $D$, and let $\mathcal{K}$ be the conjugacy class of $G$ which contains $g$. We can show that

\[ \beta(r_D, \mathcal{K}^+) = \beta(e(\mathcal{K}^+), \mathcal{K}^+), \]

by modifying the proof of Corollary 2.3. We can then adapt the proofs of Proposition 2.5 and Theorem 2.9 to show that

\[ \beta(r_D, g) = |\{P \in \text{Syl} \mid P \cap P^g = D\}|1_F. \]

(2.11)

The number of Sylow 2-subgroups of $G$ which contain $D$ is odd, by a well known generalization of Sylow’s Theorem. It then follows from our hypothesis, and (2.11), that $\beta(r_D, g) = 1_F$. So $G$ has a real 2-block $B \leftrightarrow e \leftrightarrow \omega$ which has defect group $D$, and $\beta(e, \mathcal{K}^+) = \beta(e, g) \neq 0_F$. Also $\omega(\mathcal{K}^+) = \omega(\mathcal{K}^{o+}) \neq 0_F$, by Lemma 2.1. This completes the proof.

\[ \square \]
3. REAL 2-REGULAR CLASSES AND 2-BLOCKS

In this section we prove Theorem 3.1 and give a number of applications.

Proof of Theorem 3.1. Recall that \( g \) is a 2-regular element of \( G \) with defect pair \( (D, E) \). Note that if \( E \leq P \), then \( PgP = Pg^{-1}P \).

We claim that \( E \) acts on the set \( \phi(g) := \{ P \in Syl \mid P \cap P^g = D, PgP = Pg^{-1}P \} \) by conjugation. For, suppose that \( P \in \phi(g) \). Then \( D \) normalizes \( P \). If \( e \in E \setminus D \) then \( g^e = g^{-1} \). So \( P^e g P^e = (Pg^{-1}P)^e = (PgP)^e = P^e g^{-1}P^e \).

Moreover \( eg = g^{-1}e \) normalizes \( D \). Thus \( P^e \cap P^g = (P^g \cap P)^{g^{-1}e} = D^{g^{-1}e} = D \).

This shows that \( P^e \in \phi(g) \), which proves our claim.

Each \( E \)-orbit on \( \phi(g) \) has cardinality 1 or 2. Since \( P \) is a Sylow 2-subgroup of \( G \), it is stablized by \( E \) if and only if \( E \leq P \). We conclude that

\[
|\phi(g)| \equiv |\{ P \in Syl \mid P \cap P^g = D, E \leq P \}| \pmod{2}.
\]

The result now follows from Theorem 2.9.

\[
\square
\]

In our first application of Theorem 3.1, we give another proof of \([4, 4.8]\).

**Theorem 3.2.** Let \( D \) be a 2-subgroup of \( G \). Suppose that no subgroup of \( N(D)/D \) is isomorphic to a dihedral group of order 8. Then \( \beta(e_D, g) = 1_F \), for each real 2-regular element \( g \) of \( G \) which has defect group \( D \). In particular, the following are equivalent:

(a). \( G \) has a real 2-regular element with defect group \( D \);

(b). \( G \) has a 2-block with defect group \( D \);

(c). \( G \) has a real 2-block with defect group \( D \).

Proof. The implications (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) follow as in \([4, 4.8]\).
Suppose that $D$ is a Sylow 2-subgroup of $G$. Then the principal 2-block $B_0 \leftrightarrow e_0 \leftrightarrow \omega_0$ is the only real block with defect group $D$, and the identity class is the only real 2-regular class with defect group $D$. Also $\beta(e, 1^+) = \beta(e^p, 1^+)$, for each non-real 2-block idempotent. It follows that $\beta(e_D, 1^+) = \beta(e_0, 1^+) = \omega_0(1^+) = 1_F$, using Lemma 2.1 (this also follows from a theorem of R. Brauer).

Suppose that $D$ is not a Sylow 2-subgroup of $G$. Let $g$ be a real 2-regular element with defect pair $(D, E)$. The first statement and the implication $(a) \implies (c)$ will follow from Theorem 3.1, if we can show that $P \cap P^g = D$, whenever $P$ is a Sylow 2-subgroup of $G$ which contains $E$.

Assume for the sake of contradiction that there exists $P \in \text{Syl} \ell$ with $E \leq P$ and $P \cap P^g > D$. Let $x \in E \setminus D$, and set $y := x^{-1}g = g^{-1}x^{-1}$. Then $y \in N(D)$, since $x \in E \leq N(D)$ and $g \in C(D)$. Also $y \in N(P \cap P^g)$, since $(P \cap P^g)^y = P^{x^{-1}g} \cap P^{x^{-1}} = P^g \cap P$.

The 2-group $\langle y \rangle$ acts on the nontrivial 2-group $N_{P \cap P^g}(D)/D$. So we can choose $n \in N_{P \cap P^g}(D) \setminus D$ such that $n^2 \in D$ and $[n, y] \in D$. Our hypothesis on $N(D)/D$ forces $[n, x] \in D$. Therefore $[n, g] = [n, xy] = [n, x][n, y]^{x^{-1}}$ lies in $D$. It follows that $g$ centralizes $\langle D, n \rangle$, since $\langle g \rangle$ is a 2'-group which acts trivially on every factor of $1 \leq D < \langle D, n \rangle$. This contradicts the fact that $D$ is a defect group of $g$. The theorem follows.

\[\square\]

In our next application we give self contained proofs of a number of results on extremal 2-blocks which are due to M. Herzog [6]. We call $G$ a C1-group if every intersection of distinct Sylow 2-subgroups of $G$ is centralized by some Sylow 2-subgroup of $G$. It is straightforward to show that every subgroup and factor group of a C1-group is a C1-group. Let $S$ and $T$ be Sylow 2-subgroups of $G$. We say that $S \cap T$ is a maximal Sylow intersection in $G$ if $S \neq T$ and
whenever $S \cap T \leq P \cap Q$, where $P \neq Q$ are Sylow 2-subgroups of $G$, then $S \cap T = P \cap Q$.

**Lemma 3.3.** Let $G$ be a CI-group. Suppose that $S \neq T$ are Sylow 2-subgroups of $G$. Then $S \cap T$ is centralized by every 2-group which contains it.

**Proof.** Let $R$ be a 2-subgroup of $G$ which contains $S \cap T$. Since $S \neq T$, we may assume that $R \neq T$. Then $R \cap T \geq S \cap T$. It is no loss to assume that $R = S$ and moreover that $S \cap T$ is a maximal Sylow intersection in $G$.

Now $Z(S) \leq C(S \cap T)$. So we can find a Sylow 2-subgroup $X$ of $G$ which centralizes $S \cap T$ and contains $Z(S)$. Then $S \cap T \leq X$, since $X$ normalizes $S \cap T$. It follows that $S \cap T \leq S \cap X$. If $S = X$ we are done. So assume that $S \neq X$. Then $S \cap T = S \cap X$, as $S \cap T$ is a maximal Sylow intersection. In particular $Z(S) \leq S \cap T$. But $S \cap T \leq Z(X)$ and $|Z(S)| = |Z(X)|$. So $Z(S) = S \cap T = Z(X)$.

Here is our main result:

**Theorem 3.4.** Let $G$ be a CI-group. Then $\beta(e_D, g) = 1_F$, for each real 2-regular element $g \in G$ which has defect group $D$. In particular, the statements (a), (b) and (c) of Theorem 3.2 are equivalent.

**Proof.** The implications (c) $\implies$ (b) $\implies$ (a) follow as in Theorem 4.8 of [4].

Let $g$ be a real 2-regular element of $G$ which has defect pair $(D, E)$. Choose $s \in E \setminus D$ and set $t := sg$. Then $s$ and $t$ are 2-elements which invert $g$ and $s^2 = t^2$ lies in $D$. Let $S$ be a Sylow 2-subgroup of $G$ which contains $E$. Then $t \in N(S \cap S^g)$ since $(S \cap S^g)^t = S^{at} \cap S^{agt} = S^a \cap S^{t^2} = S^a \cap S$. So $\langle S \cap S^g, t \rangle$ is a 2-group which contains $S \cap S^g$. We deduce from Lemma 3.3 that $t$ centralizes
$S \cap S^g$. Also $s \in S$ also centralizes $S \cap S^g$, again using Lemma 3.3. So $S \cap S^g$ is a 2-subgroup of $C(g)$. It follows that $S \cap S^g = D$, as $D \leq S \cap S^g$ and $D$ is a Sylow 2-subgroup of $C(g)$. The first statement and the implication (a) $\implies$ (c) now follow as in Theorem 3.2.

\[\square\]

We can now prove:

**Proposition 3.5.** Let $G$ be a CI-group and let $D$ be a maximal Sylow intersection in $G$. Then $G$ has a real 2-block with defect group $D$.

**Proof.** Note that $D$ is the largest normal 2-subgroup of $N(D)$, and also that it is not a Sylow 2-subgroup of $N(D)$.

We claim that $N(D)$ has a nonidentity real 2-regular element. For suppose otherwise. Then $N(D)/D$ has no nonidentity real 2-regular elements. It follows from the Baer-Suzuki theorem that $N(D)/D$ has a nontrivial normal 2-subgroup, which contradicts the first paragraph.

Theorem 3.4 now shows that $N(D)$ has a real 2-block $b$ with non-maximal defect. But $b$ has a defect group which contains $D$, by a theorem of R. Brauer. It follows that $D$ is a defect group of $b$. The proposition now follows from Brauer’s first main theorem.

\[\square\]

Theorems 1 and 2 of [6] are consequences of the following corollaries:

**Corollary 3.6.** Let $G$ be a finite group. Then $G$ has a normal Sylow 2-subgroup if and only if $G$ is a CI-group with no real non-principal 2-blocks.

**Proof.** The ‘only if’ part is straightforward.
Suppose that $G$ is a CI-group which has no real non-principal 2-blocks. Proposition 3.5 implies that $G$ has no maximal Sylow intersections. So $G$ has a normal Sylow 2-subgroup.

We call $G$ a TI-group if every pair of distinct Sylow 2-subgroups of $G$ intersect in the identity.

**Corollary 3.7.** Let $G$ be a finite group. Then $G$ is a TI-group if and only if $G$ is a CI-group and all real non-principal 2-blocks of $G$ have defect 0.

**Proof.** The ‘only if’ part is straightforward.

Suppose that $G$ is a CI-group and all real non-principal 2-blocks of $G$ have defect 0. We may assume that $G$ does not have a normal Sylow 2-subgroup. Let $D$ be a maximal Sylow intersection in $G$. Then $G$ has a real 2-block with defect group $D$, by Proposition 3.5. It follows from the hypothesis that $D = \{1\}$. So $G$ is a TI-group.

4. **Extended defect groups for real 2-blocks**

In this section we introduce the notion of defect pairs for real 2-blocks. We defined the relation $\preceq$ in (1.1). Now $\preceq$ is almost a partial order, in the sense that if $\mathcal{K}$ and $\mathcal{L}$ are properly real classes and if $\mathcal{M}$ is a real class, then

$$\mathcal{K} \preceq \mathcal{L} \text{ and } \mathcal{L} \preceq \mathcal{M} \implies \mathcal{K} \preceq \mathcal{M}.$$ 

Also

(4.1) $\mathcal{K} \preceq \mathcal{L}$ and $\mathcal{L} \preceq \mathcal{K} \implies \mathcal{K}$ and $\mathcal{L}$ have the same defect pairs.
Set \([\mathcal{K}] := \mathcal{K} \cup \mathcal{K}^o\), for each class \(\mathcal{K}\) of \(G\), and let

\[Z^* := \sum F[\mathcal{K}]^+,\]

where \(\mathcal{K}\) ranges over the classes of \(G\). Then \(Z^*\) is a subalgebra of \(Z\), as it coincides with the set of fixed points of the involutory automorphism \(x \mapsto x^o\) of \(Z\). Each real 2-block idempotent of \(FG\) lies in \(Z^*\). By inspecting the proof of Theorem 2.1 of [3], we see that the following is true:

**Proposition 4.2.** Suppose that \(\mathcal{L}\) is a real class of \(G\) and that \(\mathcal{K}^+\) is a properly real class which lies in the ideal of \(Z^*\) generated by \(\mathcal{L}^+\). Then \(\mathcal{K} \not\succeq \mathcal{L}\).

R. Gow showed in [3, 1.2] that if \(B \leftrightarrow e \leftrightarrow \omega\) is a real 2-block of \(G\), then there exists a real 2-regular class \(\mathcal{K}\) of \(G\) such that \(\beta(e, \mathcal{K}^+) \neq 0\) and \(\omega(\mathcal{K}^+) \neq 0\). He called any such class a real defect class for \(B\). We will call the defect pairs of the real defect classes of \(B\) the **defect pairs** of \(B\).

**Proof of Theorem 4.3.** Suppose that \(\omega(\mathcal{K}^+) \neq 0\). Then \(e = \omega(\mathcal{K}^+)^{-1} \epsilon(\mathcal{K}^+) e\). Also \(\epsilon(\mathcal{K}^+) = (\mathcal{K}^+)^{2n}\), for some integer \(n > 0\), as in Section 2. So \(e\) lies in the ideal of \(Z^*\) which is generated by \(\mathcal{K}^+\). But \(\beta(e, \mathcal{L}^+) \neq 0\). So \(\mathcal{L} \not\succeq \mathcal{K}\), by Proposition 4.2.

Suppose that \(\beta(e, \mathcal{K}^+) \neq 0\). Then, using the fact that \(\omega(\mathcal{L}^+) \neq 0\), the argument of the previous paragraph shows that \(\mathcal{K} \not\succeq \mathcal{L}\).

}\(\square\)

Let \(P, Q, R\) and \(S\) be subgroups of \(G\). We say that the pairs \((P, Q)\) and \((R, S)\) are **conjugate in** \(G\) if there exists \(g \in G\) such that \(R = P^g\) and \(S = Q^g\). Our corollary is an immediate consequence of (4.1) and Theorem 4.3:

**Corollary 4.4.** The defect pairs of a real 2-block are conjugate in \(G\).
5. The number of real 2-blocks with a given defect pair

We begin this section with a result which associates a certain $S$-orbit of 2-groups to each self-dual $(S, S)$-double coset.

**Lemma 5.1.** Suppose that $x \in G \setminus S$ and that $SxS = Sx^{-1}S$. Set $(S \cap S^g)^* := (S \cap S^g) \cup (Sg \cap g^{-1}S)$, for each $g \in SxS$. Then $(S \cap S^g)^*$ is a 2-subgroup of $G$ which contains $S \cap S^g$ as a subgroup of index 2. Moreover the $(S \cap S^g)^*$ forms a single $S$-conjugation orbit, and $Sg \cap g^{-1}S$ coincides with the set $\{ y \in Sg : y^2 \in S \cap S^g \}$.

**Proof.** First we show that $Sg \cap g^{-1}S$ is nonempty. We may write $g^{-1} = sgt$, for certain $s, t \in S$. Then $sg = g^{-1}t$ is an element of $Sg \cap g^{-1}S$.

We claim that $Sg \cap g^{-1}S$ is a right $(S \cap S^g)$-coset. Let $a = b^g \in S \cap S^g$, where $a, b \in S$, and let $cg = g^{-1}d \in Sg \cap g^{-1}S$, where $c, d \in S$. Then $(cg)a$ also lies in $Sg \cap g^{-1}S$ since $g^{-1}(da) = (cg)a = (cb)g$. Also $(cg)(sg)^{-1}$ lies in $S \cap S^g$ since $cs^{-1} = (cg)(g^{-1}s^{-1}) = (g^{-1}d)(t^{-1}g) = (dt^{-1})^g$. This proves our claim.

Now $sg \in N(S \cap S^g)$, since $(S \cap S^g)^{S^g} = S^{S^g} \cap S^g = S^g \cap S$. It follows from this and the previous paragraphs that $(S \cap S^g)^*$ is a subgroup of $G$, which contains $S \cap S^g$ as a subgroup of index 2.

Write $g = u xv$, where $u, v \in S$. Then it is clear that $(S \cap S^g)^* = (S \cap S^v)^*$. So the 2-groups $(S \cap S^g)^*$ form a single $S$-orbit of subgroups of $G$.

Finally, suppose that $y = zg$, for $z \in S$. If $y^2 \in S \cap S^g$, then $y^2 = u^g$, for some $u \in S$. Thus $y = (g^{-1}ug)(g^{-1}z) = g^{-1}uz$ lies in $Sg \cap g^{-1}S$. Conversely, suppose that $y \in Sg \cap g^{-1}S$. Then $y = g^{-1}v$, for some $v \in S$. Hence $zv = y^2 = (vz)^g$ lies in $S \cap S^g$. This proves the last statement of the lemma.

\qed
The subgroups \((S \cap S^x)^*\) have appeared in the literature on self-inverse double cosets. See for example 12.13.(ii) of [2].

Let \(\mathcal{K}\) and \(\mathcal{L}\) be 2-regular conjugacy classes of \(G\), which have defect groups \(D\) and \(Q\) respectively. Let \(x\) be any element of \(G\). The 2-group \(S \cap S^x\) acts by conjugation on \(\mathcal{K} \cap Sx\) and on \(\mathcal{L} \cap Sx\). So \(|\mathcal{K} \cap Sx| \equiv |C(\mathcal{K}, x)| (\text{mod} 2)\), where \(C(\mathcal{K}, x) := \mathcal{K} \cap Sx \cap C(S \cap S^x)\). Let \(O(\mathcal{L}, x)\) denote the set of those orbits of \(S \cap S^x\) on \(\mathcal{L} \cap Sx\) which have a representative \(l\) such that \(S \cap S^x\) contains a Sylow 2-subgroup of \(C(l)\). Now \(|\mathcal{L} \cap Sx| = \sum |S \cap S^x : C_{S \cap S^x}(l)|\). So \(|Q|/|\mathcal{L} \cap Sx|/|S \cap S^x|\) is an integer which has the same parity as \(|O(\mathcal{L}, x)|\).

We will use \(\sum_{x}^{Q \leq D}\) denote a sum which ranges over those double cosets \(SxS\) for which \(Q \leq_G S \cap S^x \leq_G D\), and \(\sum_{x \equiv x^{-1}}^{Q \leq D}\) to denote the restriction of this sum to the self-dual double cosets. These sums are empty unless \(Q \leq_G D\). It follows from [12, 1.3.3 and 1.3.4], and the fact that \(\text{Char}(F) = 2\), that

\begin{equation}
\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_{x}^{Q \leq D} |C(\mathcal{K}, x)| |O(\mathcal{L}, x)| 1_F.
\end{equation}

Let \(C^*(\mathcal{K}, x)\) denote the set of elements of \(C(\mathcal{K}, x)\) which are inverted by some element of \((S \cap S^x)^*\), and let \(O^*(\mathcal{L}, x)\) denote the set of orbits in \(O(\mathcal{L}, x)\) whose elements are inverted by some element of \((S \cap S^x)^*\).

**Proposition 5.3.** Suppose that \(\mathcal{K}\) and \(\mathcal{L}\) are real 2-regular classes of \(G\), with defect groups \(D\) and \(Q\) respectively. Then

\[\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_{x \equiv x^{-1}}^{Q \leq D} |C^*(\mathcal{K}, x)| |O^*(\mathcal{L}, x)| 1_F.\]
Proof. By pairing each double coset in (5.2) with its dual, as in the proof of Proposition 2.5, we see that

$$\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_{x \in x^{-1}} |C(\mathcal{K}, x)| |O(\mathcal{L}, x)| 1_F.$$ 

Suppose that $S_x S = S_x^{-1} S$, where $x \in G$. Let $s x \in S_x$ and $t x = x^{-1} u \in S_x \cap x^{-1} S$, where $s, t, u \in S$. Then $(s x)^{-1} t x = (u^{-1} x)(x^{-1} s^{-1})(t x) = u^{-1} s^{-1} t x$ also lies in $S_x$. Set

$$y \cdot z := \begin{cases} y^z, & \text{if } z \in S \cap S^x; \\ (y^{-1})^z, & \text{if } z \in (S \cap S^x)^* \backslash (S \cap S^x). \end{cases}$$

for each $y \in S_x$ and $z \in (S \cap S^x)^*$. It is straightforward to show that this defines an action of the 2-group $(S \cap S^x)^*$ on $S_x$.

Now $(S \cap S^x)^*$ stabilizes $C(\mathcal{K}, x)$, and also each $S \cap S^x$-orbit in $O(\mathcal{L}, x)$. Since $(S \cap S^x)^*$ is a 2-group, this implies that

$$|C(\mathcal{K}, x)| \equiv |C^*(\mathcal{K}, x)| \pmod{2} \quad \text{and} \quad |O(\mathcal{L}, x)| \equiv |O^*(\mathcal{L}, x)| \pmod{2}.$$ 

The proposition follows from this.

Proof of Theorem 5.2. Recall the notation established in Section 1.

Let $B_1 \leftrightarrow e_1 \leftrightarrow \omega_1$, \ldots, $B_u \leftrightarrow e_u \leftrightarrow \omega_u$, be a complete list of the (real) 2-blocks of $G$ which have defect pair $(D, E)$. Suppose that $1 \leq i, j \leq u$. Then (5.5)

$$\delta_{ij} = \omega_j(e_i) = \sum \beta(e_i, \mathcal{K}^+) \omega_j(\mathcal{K}^+),$$

where $\mathcal{K}$ runs through the conjugacy classes of $G$. Suppose that $\mathcal{K} \neq \mathcal{K}^o$. Then the contribution of $\mathcal{K}$ and $\mathcal{K}^o$ to (5.5) is

$$\beta(e_j, \mathcal{K}^+) \omega_i(\mathcal{K}^+) + \beta(e_j, \mathcal{K}^o^+) \omega_i(\mathcal{K}^o^+) = 2 \beta(e_j, \mathcal{K}^o^+) \omega_i(\mathcal{K}^o^+) = 0.$$
Also any real class which occurs with non-zero multiplicity in (5.5) is 2-regular and is not the trivial class. So any such class is properly real. It follows from Theorem 4.3 that

\[(5.6) \quad \delta_{ij} = \sum_{k=1}^{v} \beta(e_i, \mathcal{K}_k^+) \omega_j(\mathcal{K}_k^+).\]

Form the \(u \times v\)-matrices \(A\) and \(B\) by setting the \(i, j\)-th entry of \(A\) as \(A_{ij} = \beta(e_i, \mathcal{K}_j^+)\) and the \(i, j\)-th entry of \(B\) as \(B_{ij} = \omega_i(\mathcal{K}_j^+)\). Then \(AB^T\) is the \(u \times u\) identity matrix, by (5.6). It follows that the \(v \times v\)-matrix \(B^T A\) has rank \(u\).

Suppose that \(B = e = \omega\) is a non-real 2-block of \(G\) and that \(1 \leq i, j \leq v\). Then, since \(\mathcal{M}_i = \mathcal{K}_i^\circ\) and \(\mathcal{M}_j = \mathcal{K}_j^\circ\), the contribution of \(e\) and \(e^o\) to \(\beta(\mathcal{K}_i^+, \mathcal{K}_j^+)\) is

\[\omega(\mathcal{K}_i^+) \beta(e, \mathcal{K}_j^+) + \omega^o(\mathcal{K}_i^+) \beta(e^o, \mathcal{K}_j^+) = 2 \cdot \omega(\mathcal{K}_i^+) \beta(e, \mathcal{K}_j^+) = 0.\]

Thus

\[\beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+) = \beta \left( \sum \omega(\mathcal{K}_i^+) e, \mathcal{K}_j^+ \right) = \sum \omega(\mathcal{K}_i^+) \beta(e, \mathcal{K}_j^+),\]

where \(B = e = \omega\) runs through the real 2-blocks of \(G\). So by Theorem 4.3 we have

\[\beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+) = \sum_{k=1}^{u} \omega_k(\mathcal{K}_i^+) \beta(e_k, \mathcal{K}_j^+).\]

The sum on the right hand side is the \(i, j\)-th entry of the matrix \(B^T A\). We conclude that the \(v \times v\) matrix \(M\) with \(i, j\)-th entry \(M_{ij} = \beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+)\) has rank \(u\).

It now follows from Proposition 5.3 that \(u = 0\) if \(w = 0\), and

\[M_{i,j} = \sum_{k=0}^{w} C^*(\mathcal{K}_i, x_k) C^*(\mathcal{K}_j, x_k),\]
if \( w > 0 \). But \( C^*(X_i, x_k) = N_k \) and \( C^*(X_j, x_k) = N_{jk} \). We conclude that \( M = N \cdot N^T \), which completes the proof.

\[ \square \]

**References**


