1. An $S_{n-1}$-filtration of irreducible $S_n$-modules

We study the irreducible representations of the symmetric group $\Sigma_n$ over a field $F$ of positive characteristic $p$. For convenience, but no loss of generality, we shall assume that $F$ is algebraically closed. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ be a partition of $n$. As usual the set of nodes $[\lambda] := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$ is called the Young diagram of $\lambda$. We represent $[\lambda]$ as a set of square boxes in $\mathbb{Z}^2$, by placing a square with opposite corners $(i - 1, j - 1)$ and $(i, j)$, for each $(i, j) \in [\lambda]$. In accordance with the anglo-american convention, we orient the positive direction of the $Y$-axis downwards. The transpose $\lambda'$ of $\lambda$ is the partition of $n$ defined by $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$, for all $i$. The Young diagram of $\lambda'$ is obtained from $[\lambda]$ by reflection in the main diagonal.

We use $[1, n]$ to denote the set of integers $\{1, \ldots, n\}$. We fix a $\lambda$-tableaux $t$ for the remainder of the paper. So $t$ is a function $[\lambda] \to [1, n]$. We let $(i, j)t$ be the image of $(i, j) \in [\lambda]$ under $t$. The transpose of $t$ is the $\lambda'$-tableau $t'$ such that $(i, j)t' = (j, i)t$, for all $(i, j) \in [\lambda']$. The group $\Sigma_n$ acts on $\lambda$-tableau by permuting the contents of the boxes in a tableau. Thus $(i, j)(t\pi) = ((i, j)t)\pi$, for all $(i, j) \in [\lambda]$ and $\pi \in \Sigma_n$. If $t$ is a bijection, we let $it$ denote the node of $[\lambda]$ occupied by $i$, for
\(i = 1, \ldots, n\). Then \(i(t\pi) = (i\pi^{-1})t\). If \(s\) is another bijective \(\lambda\)-tableau, we let \(\text{sgn}(s/t)\) be the sign of the unique permutation \(\pi \in \Sigma_n\) such that \(s = t\pi\).

Let \(R_t\) be the row stabilizer, and let \(C_t\) be the column stabilizer, of \(t\) in \(\Sigma_n\). So \(R_t \cong \Sigma_\lambda\) and \(C_t \cong \Sigma_{\lambda'}\) are Young subgroups of \(\Sigma_n\). The relation \(t_1 \sim t_2\) if \(t_2 = t_1\pi\), for some \(\pi \in R_{t_1}\), defines an equivalence relation on \(\lambda\)-tableau. The equivalence class of \(\sim\) that contains \(t\) is denoted by \(\{t\}\) and is called a \(\lambda\)-tableau. The \(\mathbb{Z}\)-span of the \(\lambda\)-tableaux forms a \(\Sigma_n\)-permutation lattice \(M^\lambda\). So \(M^\lambda\) is isomorphic to the induced module \(\mathbb{Z}_{\Sigma_\lambda} \uparrow^{\Sigma_n}\).

The row stabilizer sum of \(t\) is \(R_t^+ := \sum_{\sigma \in R_t} \sigma\), while the signed column stabilizer sum of \(t\) is \(C_t^- := \sum_{\sigma \in C_t} (\text{sgn} \ \sigma) \sigma\). Both are elements of \(\mathbb{Z}\Sigma_n\). The element \(e_t := \{t\}C_t^-\) of \(M^\lambda\) is called a \(\lambda\)-polytabloid. The \(\mathbb{Z}\)-span of all \(\lambda\)-polytabloids forms the Specht sublattice \(S^\lambda\) of \(M^\lambda\). Since \(e_t\pi = e_t\), for all \(\pi \in \Sigma_n\), \(S^\lambda\) is a \(\Sigma_n\)-submodule of \(M^\lambda\).

The annihilator ideal of a polytabloid \(e_t\) in \(\mathbb{Z}\Sigma_n\) is generated by the so-called Garnir elements. The following result can be used to show that the standard polytabloids form a \(\mathbb{Z}\)-basis for \(S^\lambda\). Here \(e_t\) is standard if \((i, j)t < (k, l)t\), whenever \(i \leq k, j \leq l\) and \((i, j) \neq (k, l)\) in \([\lambda]\).

**Lemma 1.** Let \(X\) and \(Y\) be subsets of the entries in two columns, \(u\) and \(v\) respectively, of \(t\). Suppose that \(|X| + |Y|\) is greater than the length of each of the columns \(u\) and \(v\). Then \(e_t \sum_{\sigma} \text{sgn}(\sigma) \sigma = 0\), where \(\sigma\) ranges over the elements of a set of representatives \(\Sigma_X \times \Sigma_Y \backslash \Sigma_{X \cup Y}\) for the right cosets of \(\Sigma_X \times \Sigma_Y\) in \(\Sigma_{X \cup Y}\).

We say that a relation involving \(X\) and \(Y\) as above is a simple Garnir relation if \(|X| = 1\) or \(|Y| = 1\). One fact about polytabloids that is obvious from their definition, but not from the Garnir relations, is that
\( e_s = e_t \), whenever \( s \) is obtained from \( t \) by transposing two columns of equal length.

We list the removable nodes in \( [\lambda] \) as \((r_1, c_1), \ldots, (r_m, c_m)\). So for each \( u \) we have \( c_u = \lambda_{r_u} \) and \( r_u = l \) or \( \lambda_{r_u} > \lambda_{r_u+1} \). Set

\[
\lambda \mid u := (\lambda_1 \geq \cdots \geq \lambda_{u-1} \geq \lambda_u - 1 \geq \lambda_{u+1} \geq \cdots \geq \lambda_l).
\]

So \([\lambda \mid u] \) is obtained by removing the node \((r_u, c_u)\) from \([\lambda] \). For convenience we set \( r_0 := 0 \) and \( c_{m+1} := 0 \). The addable nodes of \([\lambda] \) are then \( \{(r_u + 1, c_u + 1) \mid u = 0, \ldots, m\} \). The residue of the \( u \)-th removable node is the integer \( \alpha_u := c_u - r_u \) and the residue of the \( u \)-th addable node of \( \lambda \) is \( \beta_u := c_{u+1} - r_u \). The \( p \)-residues are obtained by considering these residues mod \( p \).

As \( M^\lambda \) and \( S^\lambda \) are \( \mathbb{Z} \)-lattices, we can define \( F\Sigma_n \)-modules \( M_F^\lambda := M^\lambda \otimes_{\mathbb{Z}} F \) and \( S_F^\lambda := S^\lambda \otimes_{\mathbb{Z}} F \). There is a \( \Sigma_n \)-invariant symmetric bilinear form defined on \( M^\lambda \) such that \( \langle \{t_1\}, \{t_2\} \rangle = \delta_{\{t_1\}\{t_2\}} 1_F \), for all \( \lambda \)-tabloids \( \{t_1\} \) and \( \{t_2\} \). The restriction of this form to \( S_F^\lambda \) is generally degenerate. Given \( X \subseteq S^\lambda \), we use \( X^\perp \) to denote the dual space \( \{y \in M^\lambda \mid \langle x, y \rangle = 0, \forall x \in X\} \). Set \( J^\lambda = (S^\lambda)^\perp \cap S^\lambda \). Then by James [5] the quotient \( D_F^\lambda := S_F^\lambda / J_F^\lambda \) is either zero or an irreducible \( F\Sigma_n \)-module. The former case occurs precisely when \( \lambda \) is \( p \)-regular: that is, when no part of \( \lambda \) is repeated \( p \) or more times. Moreover \( \{D_F^\lambda \mid \lambda \) is a \( p \)-regular partition of \( n\} \) is a complete set of representatives for the isomorphism classes of irreducible \( F\Sigma_n \)-modules.

The sign \( \Sigma_n \)-module is the 1-dimensional Specht module \( S^{[1^n]} \). Let \( t \) be the \([1^n]\)-tableau such that \((i, 1)t = i \), for \( i = 1, \ldots, n \). The single vector \( \Sigma_n^t := e_t \) spans \( S^{[1^n]} \) and \( \Sigma_n^-\sigma = (\text{sgn} \sigma)\Sigma_n^\sigma \), for all \( \sigma \in \Sigma_n \).

Now set \( S^{\lambda t} := S^\lambda \otimes S^{[1^n]} \). Then \( S^{\lambda t} \) is a \( \Sigma_n \)-module with \( \mathbb{Z} \)-basis \( \{e_t \otimes \Sigma_n^-\} \), where \( t \) ranges over the standard \( \lambda^t \)-tableau. Here \( e_t \otimes \Sigma_n^- \sigma = \)
Lemma 2. There is an $\mathbb{Z}\Sigma_n$-exact sequence

$$0 \longrightarrow S^{\lambda\perp} \longrightarrow M^\lambda \xrightarrow{\theta_\lambda} S^{\lambda\ast} \longrightarrow 0,$$

such that $\{s\}\theta_\lambda := \text{sgn}(s/t) \, e_s \otimes \Sigma_n^-$, for each $\lambda$-tableau $\{s\}$. Moreover, if $\lambda$ is $p$-regular, taking images mod $p$, the above sequence restricts to

$$0 \longrightarrow J^\lambda_p \longrightarrow S^\lambda_p \xrightarrow{\theta_\lambda} D^\lambda_p \longrightarrow 0.$$

Proof. This is the substance of [5, 6.8.8.15]. Note that

$$e_s\theta_\lambda = \text{sgn}(s/t) \, e_{s'} R^{+}_{s'} \otimes \Sigma_n^-, \quad \text{for each } \lambda\text{-tableau } s.$$

If $X(\lambda)$ is any region of $[\lambda]$ then $X \, t$ will denote the image of $X$ under $t$. So $X \, t$ is the set of integers that occupy the nodes of $X(\lambda)$ in $t$.

We give some names to various regions of $[\lambda]$. The rim of $[\lambda]$ is the set of nodes $\text{Rim}(\lambda) := \{(i, \lambda_i) \mid i = 1, \ldots, l\}$. Fix $u \in [1, m]$. Let $\text{Rim}_u(\lambda)$ be the set of nodes in $\text{Rim}(\lambda)$ that belong to column $c_u$. So $\text{Rim}_u(\lambda) = \{(r_u-1+1, c_u), (r_u-1+2, c_u), \ldots, (r_u, c_u)\}$. Also $\text{Top}_u(\lambda)$ denotes the set of nodes in the top $r_u$ rows of $[\lambda]$, and $\text{Right}_u(\lambda)$ denotes the set of nodes in the right columns $c_{u+1}, \ldots, c_1$ of $[\lambda]$.

Define

$$M^\lambda_u := \mathbb{Z}\text{-span} \{t \mid n \in \text{Top}_u \, t\}.$$

Then $M^\lambda_u$ is a $\mathbb{Z}\Sigma_{n-1}$-submodule of $M^\lambda$ and

$$M^\lambda = M^\lambda_m \supset M^\lambda_{m-1} \supset \ldots M^\lambda_1 \supset M^\lambda_0 = 0.$$
is a filtration of $M^\lambda$, as $\Sigma_{n-1}$-module. Moreover $S^\lambda_u := M^\lambda_u \cap S^\lambda$ coincides with $\mathbb{Z}$-span \{ $e_t \mid t$ is a standard $\lambda$-tableau and $n \in \text{Right}_u t$ \}. Thus (as in [5]) $S^\lambda|_{\Sigma_{n-1}}$ has a Specht filtration

$$S^\lambda = S^\lambda_m \supset S^\lambda_{m-1} \supset \ldots \supset S^\lambda_1 \supset S^\lambda_0 = 0,$$

Each $S^\lambda_u$ is an $\Sigma_{n-1}$-submodule of $S^\lambda|_{\Sigma_{n-1}}$ and $S^\lambda_u/S^\lambda_{u-1} \cong S^\lambda_v$.

For each $u$ there is an $\Sigma_{n-1}$-exact sequence

$$0 \longrightarrow S^\lambda_{u-1} \longrightarrow S^\lambda_u \longrightarrow S^\lambda_u \longrightarrow 0,$$

with $\theta_u$ calculated as follows. If $e_t \in S^\lambda_u \setminus S^\lambda_{u-1}$ then $e_t \theta_u = \pm e_s$. Here $s$ is the $\lambda_u$-tableau that is obtained from $t$ by transposing, if necessary, $n$ with the entry at the top of its column, interchanging this column with column $\lambda_u$, and finally removing $n$. The sign is $+1$ if $n$ was at the top of its column in $t$, and $-1$ otherwise. For example, if $\lambda = [3^2, 1]$ then

$$
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
$$

$\theta_1 = -1$

Suppose now that $\lambda$ is $p$-regular. Identify $D^\lambda$ with $S^\lambda \theta_\lambda$. The restricted module $D^\lambda|_{\Sigma_{n-1}}$ has a filtration

$$D^\lambda = D^\lambda_m \supset D^\lambda_{m-1} \supset \ldots \supset D^\lambda_1 \supset D^\lambda_0 = 0,$$

where $D^\lambda_u = S^\lambda_u \theta_\lambda$. Thus $D^\lambda_u/D^\lambda_{u-1} = (S^\lambda_u + J^\lambda)/(S^\lambda_u + J^\lambda)$. Notice that this coincides with $S^\lambda_u/(S^\lambda_u + J^\lambda \cap S^\lambda_u)$, which is a quotient of the Specht module $S^\lambda$. While the filtration $\{S^\lambda_u\}$ of $S^\lambda$ is strictly decreasing, the filtration $\{D^\lambda_u\}$ of $D^\lambda$ is generally only non increasing.

Set $L_n := (1, n) + (2, n) + \ldots + (n-1, n)$ as the $n$-th Jucys-Murphy element in $\mathbb{Z} \Sigma_n$. Then $L_n$ commutes with every element of $\Sigma_{n-1}$. It follows that $L_n$ acts as $\Sigma_{n-1}$-endomorphism on every $F \Sigma_{n-1}$-module, in particular on the Specht module $S^\lambda$ and on the radical $J^\lambda$ of the
bilinear form on $S^\lambda$. If $\lambda$ is $p$-regular, then $L_n$ acts on the irreducible $S_n$-module $D^\lambda$. We aim to show that in this case $L_n$ generates the ring $\text{End}_{S_{n-1}}(D^\lambda)$ of all $\Sigma_{n-1}$-endomorphisms of $D^\lambda$.

2. Symmetric functions

We begin with some results on symmetric functions. These will be required in order to evaluate some inner product expressions obtained in later sections of the paper. The most complicated result is Theorem 5. The proof was emailed to me by Grant Walker. Our original proof was by defining an involution on certain monomials, and cancellation.

Fix positive integers $u > v$. We use the following notation:

- $\binom{u}{v}$ is the collection of subsets of $[1, u]$ of size $v$;
- $\langle u \rangle_v$ is the collection of multi-subsets of $[1, u]$ of size $v$.

We regard $\binom{u}{v}$ as the collection of decreasing functions, and $\langle u \rangle_v$ as the collection of nonincreasing functions, $[1, v] \to [1, u]$.

Let $X := \{x_1, x_2, \ldots\}$ and $Y := \{y_1, y_2, \ldots\}$ be sets of variables that are finite. Recall the elementary and complete symmetric functions of degree $n$ are

\[
E_n(Y) := \sum_{\mu \in \binom{\{y\}}{n}} y_{\mu_1} \cdots y_{\mu_n};
\]
\[
H_n(X) := \sum_{\mu \in \langle \{x\} \rangle_n} x_{\mu_1} \cdots x_{\mu_n}.
\]

We define the symmetric function $HE_n$ by

\[
HE_n(X; Y) := \sum_{i=0}^{n} (-1)^i H_{n-i}(X) E_i(Y).
\]

Then the generating function for $HE_n$ is

\[
\sum_{k \geq 0} HE_k(X; Y) t^k = \prod_{i=1}^{1} \left(1 - x_i t\right)^{-1} \prod_{j=1}^{1} \left(1 - y_j t\right).
\]
Lemma 3. Let $n, i, j > 0$. Then
\[ \text{HE}_n(X; Y) = (x_i - y_j)\text{HE}_{n-1}(X; Y\setminus\{y_j\}) + \text{HE}_n(X\setminus\{x_i\}; Y\setminus\{y_j\}). \]

Proof. Considering the generating function of $\text{HE}_n$, this follows from
\[ \frac{(1 - y_j t)}{(1 - x_i t)} = \frac{t(a - y_j)}{(1 - x_i t)} + 1. \]

Let $u, v$ be integers. There is a bijection $\mu \in \binom{u}{v}$ to $\tilde{\mu} \in \binom{u+v-1}{v}$, where $\tilde{\mu}_i := \mu_i + i - 1$, for $i = 1, \ldots, v$. This is the basis of the following lemma.

Corollary 4. Suppose that $|X| < |Y|$. Set $n := |Y| - |X| + 1$. Then
\[ \text{HE}_n(X; Y) = \sum_{\mu \in \binom{|X|}{n}} \prod_{i=1}^{n}(x_{\mu_i} - y_{\mu_i+1}). \]

Proof. For each $u, v$ define $f(u, v) := \sum_{\mu \in \binom{u}{v}} \prod_{i=1}^{v}(x_{\mu_i} - y_{\mu_i+1})$. Set $m := |X|$. The $\mu \in \binom{m}{n}$ with $\mu_n = m$ contribute $(x_{|X|} - y_{|Y|})f(m, n-1)$ to $f(m, n)$. The $\mu \in \binom{m}{n}$ with $\mu_n < m$ contribute $f(m-1, n)$ to $f(m, n)$. Thus
\[ f(m, n) = (x_{|X|} - y_{|Y|})f(m, n-1) + f(m-1, n). \]

The result now follows by induction from the previous lemma. \hfill \Box

Let $X, Y$ be sets of indeterminates, as before, and let $Z = \{z_1, z_2, \ldots\}$ be another set of indeterminates. Suppose that $0 < n \leq m$ and that $\mu \in \binom{m}{n}$. Set $\mu_0 := m$, and for $i \in [1, n]$ define
\[ \delta_{\mu,i} := \begin{cases} y_{\mu_i}, & \text{if } \mu_{i-1} < \mu_i; \\ z_i, & \text{if } \mu_{i-1} = \mu_i; \end{cases} \]
for $\mu_{-1} := 0$. So $\delta_\mu$ keeps track of the places where $\mu$ decreases.
Theorem 5. Suppose that $|X| = |Y| + 1$. Set $n := |Z|$. Then

$$\text{HE}_n(X; Y \cup Z) = \sum_{\mu \in \binom{|X|}{n}} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

Proof. For each $u, v$ define $f(u, v) := \sum_{\mu \in \binom{|v|}{u}} \prod_{i=1}^{u} (x_{\mu_i} - \delta_{\mu,i})$. Set $m := |X|$. We compare $f(m, n)$ and $f(m-1, n)$. The $\mu \in \binom{m}{n}$ with $\mu_n < m - 1$ contribute the same term to $f(m, n)$ and $f(m-1, n)$. The $\mu \in \binom{m}{n}$ with $\mu_n = m - 1$ contribute $(x_{m-1} - y_{m-1})f(m-1, n-1)$ to $f(m, n)$ and $(x_{m-1} - z_{n})f(m-1, n-1)$ to $f(m-1, n)$. The $\mu \in \binom{m}{n}$ with $\mu_n = m$ contribute $(x_{m} - z_{n})f(m, n-1)$ to $f(m, n)$ and nothing to $f(m-1, n)$. Thus we have a recursion relation

$$f(m, n) - f(m-1, n) = (z_{n} - y_{m-1})f(m-1, n-1) + (x_{m} - z_{n})f(m, n-1).$$

But $\text{HE}_n(X; Y \cup Z)$ satisfies the same recursion relation and initial conditions, as follows from examination of its generating function and the equality

$$\frac{(1 - y_{m-1}t)(1 - z_{n}t)}{(1 - x_{m}t)} - (1 - z_{n}t) = t(z_{n} - y_{m-1}) + \frac{t(x_{m} - z_{n})(1 - y_{m-1}t)}{(1 - x_{m}t)}.$$

$\square$

Corollary 6. Suppose that $|X| = |Y| + 1$ and that $Z$ is a subset of $X$ of size $n$. Then

$$\text{HE}_n(X \setminus Z; Y) = \sum_{\mu \in \binom{|X\setminus Z|}{n}} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

Proof. It follows from Theorem 5 that

$$\text{HE}_n(X; Y \cup Z) = \sum_{\mu \in \binom{|X|}{n}} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

The result now follows from Lemma 3. $\square$
3. The action of $L_n$ on a Specht filtration of $S^\lambda S_{n-1}$

Let $u$ be the largest index such that $n \in \text{Right}_u$. Equivalently $n$ occupies a column of length $r_u$ in $t$. Set $\alpha_t := c_u - r_u$ as the residue of the $u$-th removable node of $t$. Define a transitive relation $\rightarrow$ on $[1, n]$, such that $i \rightarrow j$ if $i$ occupies a longer column than $j$ in $t$. G. E. Murphy proved the next result for standard polytabloids in [8, 3.3].

Lemma 7.

$$e_t L_n = \alpha_t e_t + \sum_{n \rightarrow w} e_t(n, w).$$

In particular if $m \geq u \geq v \geq 1$ then $S^\lambda_u \prod_{w=v+1}^{u} (L_n - \alpha_w) \subseteq S^\lambda_v$.

Proof. Let $u$ be the largest index such that $n \in \text{Right}_u$ and suppose that $n$ occupies column $c$ in $t$. Then $c \leq c_u$, and column $c$ has length $r_u$. Let $1 \leq d \leq c_u$ with $d \neq c$. Then column $d$ of $[\lambda]$ has length $\geq r_u$. By a simple Garnir relation we have $\sum \{e_t(n, w) \mid w \text{ belongs to column } d \text{ of } t\} = e_t$. There are $c_u - 1$ such columns. Now $e_t(n, w) = -e_t$, if $w$ belongs to column $c$ in $t$ and $w \neq n$. There are $r_u - 1$ such integers $w$. Combining these facts we get

$$\sum \{e_t(n, w) \mid n \not\rightarrow w \text{ and } n \neq w\} = \alpha_u e_t.$$

The first statement now follows from the fact that $e_t L_n = \sum_{w=1}^{n-1} e_t(n, w)$.

The last statement follows from the first by induction. \hfill \square

We define a $t$-cycle to be any cyclic permutation in $\Sigma_n$ of the form $(n, w, x, \ldots, y)$, where

$$n \rightarrow w \rightarrow x \rightarrow \ldots \rightarrow y.$$ 

Clearly if $\pi_1$ is a $t$-cycle and $\pi_2$ is a $t\pi_1$-cycle then $\pi_1 \pi_2$ is a $t$-cycle.
We set $\overline{t}$ as the $\lambda$-tableau got by reversing the rows of $t$ i.e.

$$(i, j)\overline{t} = (i, \lambda_i - j + 1), \quad \text{for } (i, j) \in [\lambda].$$

Let $\varepsilon_i$ denote the number of parts of $\lambda$ that equal $i$. G. James proved that $S^\lambda/J^\lambda \neq 0$ if $\lambda$ is $p$-regular, by observing that

$$<e_t, e_T> = \prod_{i=1}^{n} (\varepsilon_i!)^t.$$

**Lemma 8.** Suppose that $n \in \text{Rim}_t$, and let $\pi$ be a $t$-cycle. Then

$$<e_{t\pi}, e_T> = \begin{cases} 
\text{sgn } \pi <e_t, e_T>, & \text{if } \pi \text{ fixes each integer not in Rim}_t; \\
0, & \text{otherwise.} 
\end{cases}$$

**Proof.** Suppose that $\pi$ fixes each integer not in Rim$t$. Then $\pi$ is a permutation of the first column of $\overline{t}$ and $e_T = \text{sgn}(\pi) e_{\overline{t}}$. We conclude from (2) that $<e_{t\pi}, e_T> = \text{sgn} \pi <e_t, e_T>$. Conversely, suppose that $w \pi \neq w$, for some integer $w \notin \text{Rim}_t$. Let $w$ belong to a column of length $r_u$ in $t$. Set $X_t := \{n\} \cup \text{Right}_u \cap \text{Rim}_t$. Now $\pi$ fixes each integer in $\text{Rim}_u t$ and $\text{Right}_{u-1} t \subseteq \text{Right}_u t \pi$, as $\pi$ is a $t$-cycle. It follows that $X_t \subseteq \text{Right}_u t \pi$. Let $\rho$ be a column permutation of $t \pi$ and let $\sigma$ be a column permutation of $\overline{T}$. The $r_u + 1$ integers in $X_t$ are constrained to the first $r_u$ rows of $t \pi \rho$. So two or more of them belong to the same row of $t \pi \rho$. However the first column of $\overline{T}$ coincides with Rim$t$, and thus contains $X_t$. So the elements of $X_t$ belong to different rows of $\{\overline{T} \sigma\}$. This shows that $\{\overline{T} \sigma\} \neq \{t \pi \rho\}$. We conclude that $<e_{t\pi}, e_T> = 0.$

We call a $t$-cycle $\pi$ visible if $<e_{t\pi}, e_T> \neq 0$. Lemma 8 implies that $\pi$ is visible if and only if $\pi$ is a permutation of Rim$t$. Any $t$-cycle that is not visible is said to be invisible.
Lemma 9. Suppose that $\pi$ is invisible. Then $e_\pi L_n$ is a sum of polytabloids $e_\sigma$, where each $\sigma$ is invisible.

Proof. This follows from Lemma 7 and the observation that if $\rho$ is a $t\pi$-cycle then $\pi\rho$ is an invisible $t$-cycle. \hfill \Box

We now set $S^\lambda_t$ as the $\mathbb{Z}$-span of all $e_{t\pi}$, where $\pi$ is a $t$-cycle. Also set $I^\lambda_t$ as the sum of the following two subspaces of $S^\lambda_t$: the first is the $\mathbb{Z}$-span of all $e_{t\pi}$, where $\pi$ is an invisible $t$-cycle; the second is the $\mathbb{Z}$-span of all $(\text{sgn} \pi_1)e_{t\pi_1} - (\text{sgn} \pi_2)e_{t\pi_2}$, where $\pi_1$ and $\pi_2$ are visible $t$-cycles such that $n \in \text{Rim}_v t\pi_i$, for $i = 1, 2$, for some $v$.

Suppose that $n$ belongs to $\text{Rim}_u t$. Set $t_u := t$ and $e_u := e_t$. For $i = 1, \ldots, u - 1$, define $e_i := -e_t(n, (r_i, e_i)t)$. Clearly

\begin{equation}
e_u, e_{u-1}, \ldots, e_1 \quad \text{form a basis for } S^\lambda_t \text{ modulo } I^\lambda_t.
\end{equation}

Now identify each $e_i$ with its image in $S^\lambda_t/I^\lambda_t$.

Lemma 10. There is a well-defined action of $L_n$ on $S^\lambda_t/I^\lambda_t$. The matrix of $L_n$ with respect to $e_u, e_{u-1}, e_{u-2}, \ldots, e_2, e_1$ is

\[
\begin{pmatrix}
\alpha_u & (\alpha_{u-1} - \beta_{u-1}) & (\alpha_{u-2} - \beta_{u-2}) & \cdots & (\alpha_2 - \beta_2) & (\alpha_1 - \beta_1) \\
0 & \alpha_{u-1} & (\alpha_{u-2} - \beta_{u-2}) & \cdots & (\alpha_2 - \beta_2) & (\alpha_1 - \beta_1) \\
0 & 0 & \alpha_{u-2} & \cdots & (\alpha_2 - \beta_2) & (\alpha_1 - \beta_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_2 & (\alpha_1 - \beta_1) \\
0 & 0 & 0 & \cdots & 0 & \alpha_1
\end{pmatrix}
\]

Proof. Fix $i = 1, \ldots, u$. It follows from Lemma 7 that

\[
e_i L_n = \alpha_i e_i + \sum_{j=1}^{i-1} \sum_{w_j} e_i(n, w_j), \quad \text{modulo } I^\lambda_t.
\]
Here \( w_j \) ranges over all integers in \( \text{Rim}_j t \). But \( \text{Rim}_j \) contains \( r_j - r_{j-1} + 1 = \beta_j - \alpha_j \) nodes. The result now follows from the fact that \( e_i(n, w_j) = -e_j \) modulo \( I^\lambda_t \).

Let \( v \) be a positive integer. Specialize sets of variables \( X \) and \( Y \) such that \( x_i := \alpha_i \), for \( i \in [1, u] \) and \( y_j := \beta_j \), for \( j \in [1, u-1] \). Let \( Z := \{z_1, \ldots, z_v\} \) be a set of variable scalars, and define \( \delta_\mu \) as in (1). Define \( F_Z := \prod_{i=1}^v (L_n - z_i) \).

**Corollary 11.** \( \langle e_t F_Z, e_t \rangle = \sum_{\mu \in \binom{v}{u}} \prod_{i=1}^v (x_{\mu_i} - \delta_{\mu,i}) \).  

**Proof.** Lemma 10 and an inductive argument show that  
\[
e_t F_Z = \sum_{j=1}^u \sum_{\mu \in \binom{v}{u}} \prod_{i=1}^v (x_{\mu_i} - \delta_{\mu,i}) e_i \text{ mod } I^\lambda_t.
\]
Then Lemma 8 implies that  
\[
\langle e_t F_Z, e_t \rangle = \sum_{\mu \in \binom{v}{u}} \prod_{i=1}^v (x_{\mu_i} - \delta_{\mu,i}).
\]

4. **A lower bound on the degree of the minimal polynomial of \( L_n \) acting on \( D^\lambda \)**

It is known that the \( p \)-block of \( \Sigma_n \) that contains \( S^\lambda \) is determined by the multiset of \( p \)-residues of the nodes in \( [\lambda] \). Thus for all \( u, v \), the Specht modules \( S^{\lambda^u} \) and \( S^{\lambda^v} \) belong to the same \( p \)-block of \( \Sigma_{n-1} \) if and only if \( \alpha_u \equiv \alpha_v \) modulo \( p \). Given \( u \in [1, m] \) we use \( B_u \) to denote the
primitive idempotent in the centre of $F \Sigma_{n-1}$ that acts as the identity on $S^{\lambda_u}$. Note that the image of $B_u$ is a primitive idempotent in the centre of $\operatorname{End}_{\Sigma_{n-1}}(S^{\lambda})$.

A. Kleshchev defines the normal and good removable nodes of $\lambda$ as follows. Fix $u$. Consider the sequence $S_u := (\alpha_{u-1}, \beta_{u-1}, \alpha_{u-1}, \ldots, \alpha_1, \beta_1)$ that alternates between the residues of the addable and removable nodes of $[\lambda]$ above the $u$-th removable node. Let $T_u = (t_1, t_2, \ldots)$ be the sign sequence that is obtained from $S_u$ by removing those terms not congruent to $\alpha_u$ modulo $p$, and replacing each remaining $\alpha_i$ by $+1$ and each remaining $\beta_i$ by $-1$. Then the $u$-th removable node is normal if $\sum_{i=1}^{j} t_i \geq 0$, for each $j \geq 1$. The $u$-th removable node is good if it is the lowest normal node with $p$-residue $\alpha_u$.

Now set $\varepsilon_u$ as the number of $i \in [1, u-1]$ such that the $i$-th removable nodes of $[\lambda]$ is normal and $\alpha_i \equiv \alpha_u$ modulo $p$.

**Lemma 13.** If the $u$-th removable node of $[\lambda]$ is normal then

$$D_u^{\lambda}(L_n - \alpha_u)^{\varepsilon_u} B_u \neq 0.$$ 

**Proof.** We claim that $E_u := \prod_{i=1}^{u-1} (L_n - \alpha_i)$ is a unit in $\operatorname{End}_{\Sigma_{n-1}}(S_u^{\lambda}) B_u$.

For, $E_u$ annihilates $S_u^{\lambda}(1-B_{\alpha_u})$, as this module has a Specht series with successive quotients $S^{\lambda'}$ on which $(L_n - \alpha_i)$ acts as zero. Moreover by [4], the subalgebra $F[L_n]$ of $\operatorname{End}_{\Sigma_{n-1}}(S^{\lambda})$ is unital and local. The claim now follows from the fact that $E_u$ is not nilpotent, as it acts as the nonzero scalar $\prod_{i=1}^{u-1} (\alpha_u - \alpha_i)$ on the quotient $S_u^{\lambda}/S_{u-1}^{\lambda}$. So it is enough to show that $D_u^{\lambda}(L_n - \alpha_u)^{\varepsilon_u} E_u \neq 0$.

Assume that $n$ occupies the $u$-th removable node in $t$. Let $\sigma$ be the set of indices $i \in [1, u-1]$ such that either the $i$-th removable node of $[\lambda]$ is normal or its $p$-residue is different to $\alpha_u$. Suppose that $\sigma$ has
cardinality \( v \). So \( \sigma \in \binom{u-1}{v-1} \). Now specialize sets of variables \( X, Y\) and \( Z \) such that \( x_i := \alpha_i \), for \( i \in [1,u] \) and \( y_j := \beta_j \), for \( j \in [1, u-1] \) and \( z_k := \alpha_{\sigma_k} \), for \( k = [1,v] \). Then \( (L_n - \alpha_u)^{z_u} E_u \) coincides with \( Z_F \), in the notation of Corollary 12. Now that Corollary implies that \( \langle e_1 F_Z, e_{-1} \rangle = \text{HE}_u(X; Y \cup Z) \). We complete the proof by showing that \( \text{HE}_u(X; Y \cup Z) \not\equiv 0 \) modulo \( p \).

As the \( u \)-th removable node is normal, there is a bijection between the addable nodes with \( p \)-residue \( \alpha_u \) above the \( u \)-th removable node and the non-normal nodes with \( p \)-residue \( \alpha_u \) above the \( u \)-th removable node. Let \( i \in [1, u-1] \). If the \( i \)-th removable node is non-normal and of \( p \)-residue \( \alpha_u \), then \( x_i \equiv y_j \equiv \alpha_u \), where the \( j \)-th addable node is the corresponding one of \( p \)-residue \( \alpha_u \). Otherwise there exists \( k \) so that \( \sigma_k = i \) and \( x_i = z_k \). In this way we get an injective map \( f : X \setminus \{x_u\} \to Y \cup Z \) such that \( x_i \equiv f(x_i) \) modulo \( p \). Now \( Y \cup Z \setminus f(X \setminus \{x_u\}) \) consists of the \( v \) elements \( Y_1 \) of \( Y \) whose values do not equal \( x_u \equiv \alpha_u \). Applying Lemma 3 repeatedly, and working modulo \( p \), we can remove the \( v \) equal pairs in the graph of \( f \) from \( \text{HE}_u(X; Y \cup Z) \). Thus

\[
\text{HE}_u(X; Y \cup Z) \equiv \text{HE}_u(\{x_u\}; Y_1) \equiv \prod_{y \in Y_1} (x_u - y) \not\equiv 0 \quad \text{modulo } p.
\]

\[\square\]

5. **An upper bound on the degree of the minimal polynomial of \( L_n \) acting on \( D^\lambda \)**

Let \( R_{t,1} \) be the group of permutations in \( R_t \) that fix each entry in the row of \( t \) that contains \( n \) and let \( R_{t,2} \) be the group of permutations in \( R_t \) that fix each entry \emph{not} in this row. So \( R_t \) is the internal direct product

\[
R_t = R_{t,1} \times R_{t,2}.
\]
We write \( \rightarrow_t \) if we need to point out the dependence of the relation \( \rightarrow \) (defined in Section 3) on \( t \). Define a subrelation \( \Rightarrow \) of \( \rightarrow \) on \([1, n]\) by \( i \Rightarrow j \) if \( i \rightarrow j \) and if \( i \) and \( j \) occupy the same row of \( t \). The motivation for the following lemma comes from considering the images of polytabloids under the map \( \theta_\lambda : S^\lambda_F \rightarrow D_\lambda^\pi \), when \( \lambda \) is \( p \)-regular. Compare it with Lemma 7.

**Lemma 14.** \( e_t R_{t,1}^+ L_n = \left( \alpha_t e_t + \sum_{n \Rightarrow i} e_t(n, i) \right) R_{t,1}^+ \).

**Proof.** As each \( \pi \in R_{t,1} \) fixes all entries in the same row as \( n \) in \( t \), \( \Rightarrow \) has the same meaning for \( t \) and \( t\pi \). Moreover, if \( n \Rightarrow i \) then \( \pi(n, i) = (n, i)\pi \). Lemma 7 gives

\[
e_t R_{t,1}^+ L_n = \alpha_t e_t R_{t,1}^+ + \sum_{\pi \in R_{t,1}} \sum_{n \Rightarrow i} e_t\pi(n, i).
\]

Our proof is completed by showing that those polytabloids \( e_t \pi(n, i) \) such that \( n \) and \( i \) belong to different rows of \( t\pi \) cancel in pairs. Fix \( \pi \in R_{t,1} \) and suppose that \( n \rightarrow_t i \) but \( n \) and \( i \) belong to different rows of \( t\pi \). Let \( j \) be the integer in \( t\pi \) in the same column as \( n \) and the same row as \( i \). Then \( j \neq i, n \), and \( \pi(i, j) \in R_{t,1} \) and \( n \rightarrow_{t\pi(i,j)} j \). Since \( e_t \pi(i, j)(n, j) = e_t \pi(n, i)(i, j) \), and \( (i, j) \) is a column permutation of \( t\pi(n, i) \), the sum \( e_t \pi(i, j)(n, j) + e_t \pi(n, i) \) is zero. The Lemma follows from this. \( \square \)

Now set \( \tau_u \) as the number of \( i \in [1, u] \) such that the \( i \)-th removable nodes of \([\lambda]\) is normal and \( \alpha_i \equiv \alpha_u \) modulo \( p \). So \( \tau_u = \varepsilon_u + 1 \), if the \( u \)-th removable node of \([\lambda]\) is normal.

**Lemma 15.** \( D_\lambda^\pi(L_n - \alpha_u)\tau_u B_u = 0 \).

**Proof.** Adopt the notation and assumptions of Lemma 13. In particular \( n \) occupies the \( u \)-th removable node of \( t \). We change the definition of
so that \( \sigma \in \binom{u}{v+1} \) is the set of indices \( i \in [1, u] \) such that either the \( i \)-th removable node of \( [\lambda] \) is normal or its \( p \)-residue is not \( \alpha_u \). Recall that \( \theta_\lambda \) is a \( \Sigma_n \)-homomorphism \( S^{\lambda} \to S^{\lambda'} \otimes S^{[r^n]} \) whose image is the irreducible module \( D^\lambda \). Now specialize the values of \( Z \) as \( z_k := \alpha_{\sigma_k} \), for \( k = [1, v+1] \). Set \( F_Z := \prod_{i=1}^{v+1}(L_n - \alpha_{\sigma_i}) \). We will show that, in the notation of Corollary 12, \( e_t F_Z \theta_\lambda = 0 \).

Since \( L_n \) is a sum of permutations in \( \Sigma_n \), it commutes with \( \theta_\lambda \). Moreover, the permutations are odd. So

\[
e_t F_Z \theta_\lambda = e_t \theta_\lambda F_Z = (e_{t'} R_{t'}^+ \otimes \Sigma^-) \prod_{i=1}^{v+1}(L_n - \alpha_{\sigma_i})
\]

\[
= e_{t'} R_{t'}^+ \prod_{i=1}^{v+1}(-L_n - \alpha_{\sigma_i}) \otimes \Sigma^-.
\]

So it is enough to show that \( e_{t'} R_{t'}^+ \prod_{i=1}^{v+1}(-L_n - \alpha_{\sigma_i}) = 0 \).

Now \( n \) occupies the \( c_u \)-th row and \( r_u \)-th column of the transpose \( t' \) of \( t \). Considering the dimensions of \( [\lambda'] \), given \( \pi \in R_{t'} \), the symbol \( n \) occupies a column of length \( r_1, r_2, \ldots, r_{u-1} \) or \( r_u \) in \( t' \pi \).

Let \( i, j \in [1, u] \), with \( i < j \), and let \( \pi \in R_{t'} \) be such that \( n \) occupies a column of length \( r_j \) in \( t' \pi \). Suppose that \( k \) is a symbol in the \( c_u \)-th row of \( t' \pi \) and in a column of length \( r_j \). So \( \pi(n, k) \in R_{t'} \). Then \( n \) occupies the same row as \( k \) in \( t' \pi(n, k) \), but a column of longer length. So \( e_{t'} \pi \) is one of the polytabloids that occurs in the expansion of \( e_{t'} \pi(n, k) R_{t',1}^+ L_n \) given by Lemma 14. Moreover, this is the only way that \( e_{t'} \pi \) can occur in the expansion of \( e_{t} \eta R_{t',1}^+ L_n \), for any \( \eta \in R_{t'} \). Note that there are \( (\beta_i - \alpha_i) \) choices for \( k \), as \([\lambda']\) has \( \beta_i - \alpha_i \) columns of length \( r_i \).

For \( i \in [1, u] \), let \( f_i \) denote the sum, in \( S^{\lambda'} \), of all polytabloids \( e_{t'} \pi \), where \( n \) occupies a column of length \( r_i \) in \( t' \pi \). Note that \( \alpha_{t' \pi} = -\alpha_i \), for each such \( \pi \). Set \( t_i := t(n, (r_i, c_i)t) \), and let \( \widehat{t_i} \) be the row reversal of \( t_i \).
Lemma 14 and the previous paragraph implies that

\[ f_i(-L_u) = \alpha_i f_i + \sum_{j=i+1}^u (\alpha_i - \beta_j) f_j. \]

Now \( e_v R_v^+ = f_1 + \ldots + f_u \) and we have

\[ (f_1 + \ldots + f_u)(-L_u) = \sum_{i=i+1}^u \left( \alpha_i + \sum_{j=1}^{i-1} (\alpha_j - \beta_j) \right) f_i. \]

Using Corollary 11 and induction, it can be seen that

\[ (f_1 + \ldots + f_u) \prod_{i=1}^{v+1} (-L_n - \alpha_i) = \sum_{i=1}^n \langle e_i, F Z, e_i^{-1} \rangle f_i. \]

For \( i \in [1, u] \), we get from Corollary 12 that

\[ \langle e_i, F Z, e_i^{-1} \rangle = \text{HE}_{v+1}(\{\alpha_i, \ldots, \alpha_1\}; \{\beta_{i-1}, \ldots, \beta_1\} \cup Z). \]

Arguing as in the proof of Lemma 3, there exists an injective map \( f_i : \{\alpha_i, \ldots, \alpha_1\} \to \{\beta_{i-1}, \ldots, \beta_1\} \cup Z \) such that \( f_i(\alpha_j) \equiv \alpha_j \mod p \), for all \( j \). Thus \( \text{HE}_{v+1}(\{\alpha_i, \ldots, \alpha_1\}; \{\beta_{i-1}, \ldots, \beta_1\} \cup Z) \equiv 0 \mod p \). The lemma follows from this.

\[ \textbf{Corollary 16.} \text{ Suppose that there are no normal removable nodes of } p\text{-residue } \alpha_u \text{ at or above the } u\text{-th removable node of } [\lambda]. \text{ Then } D^u = 0. \]

\[ \textbf{Proof.} \text{ Our hypothesis is that } \tau_u = 0. \text{ Now apply the previous lemma.} \]

Our main result here is

\[ \textbf{Theorem 17.} \text{ Suppose that the } u\text{-th removable node of } [\lambda] \text{ is good. Then } (x - \alpha_u)^{\tau_u} \text{ is the minimal polynomial of } L_n \text{ acting on } D^u B_u. \]

\[ \textbf{Proof.} \text{ By definition, the } u\text{-th removable node is the lower normal removable node of } [\lambda] \text{ that has } p\text{-residue } \alpha_u. \text{ So } \tau_u = \varepsilon_u + 1. \text{ Thus} \]

\[ \]
$D^\lambda B_u(L_n - \alpha_u)^{\tau_u - 1} \neq 0$, by Lemma 13. But $\tau_u$ equals the number of normal removable nodes of $[\lambda]$ that have $p$-residue $\alpha_u$. So $D^\lambda B_u(L_n - \alpha_u)^{\tau_u} = 0$, by Lemma 15.

6. ADDITIONAL REMARKS, UNFINISHED SECTION

The next result is well known, being a special case of results of Carter and Payne [2]. Recently several authors [7], [4] have shown that in this situation $\text{Hom}_{\Sigma_n}(S^\alpha, S^\beta)$ is 1-dimensional. Here we prove existence and give a simple algorithm to compute the image of a polytabloid under the One-Box Shift homomorphism.

**Theorem 18.** Suppose that $\alpha$ and $\beta$ are partitions of $n$ such that $[\beta]$ is obtained by removing a node from $[\alpha]$ and adding it back on in a lower row, so that the removed and added positions have the same $p$-residue. Then $\text{Hom}_{\Sigma_n}(S^\alpha, S^\beta) \neq 0$.

**Proof.** Note that $\alpha$ and $\beta$ have the same $p$-core. Also, there exists positive integers $u > v$, and a partition $\lambda$ of $n + 1$ whose diagram can be obtained by adding a node to the end of row $u$ of $[\alpha]$ or to the end of row $v$ of $\beta$.

We use the notation for the residue of $\lambda$ and the Specht filtration $\{S^\lambda_u\}$ of $S^\lambda_{\Sigma_n}$ as described above, noting that here $\lambda$ is a partition of $n + 1$, and not $n$ as before. For integers $i > j$ define $\theta_{i,j} := \prod_{i=\max(u,v) + 1}^{u}(L_{n+1} - \alpha_i)$ Then $\theta_{u,v} = \theta_{u-1,v-1}$, as $\alpha_u = \alpha_v$. So by the previous corollary, we have $S^\lambda_u \theta_{u,v} \subseteq S^\lambda_v$ and also $S^\lambda_{u-1} \theta_{u,v} \subseteq S^\lambda_{v-1}$. Moreover $S^\lambda_u \theta_{u,v} \not\subseteq S^\lambda_{v-1}$ by a result in [3]. Thus $\theta_{u,v}$ induces a non-zero $\Sigma_n$-homomorphism, denoted $f_{u,v}$, from $S^\alpha = S^\lambda_u/S^\lambda_{u-1}$ into $S^\beta = S^\lambda_v/S^\lambda_{v-1}$. As $\text{Hom}_{\Sigma_n}(S^\alpha, S^\beta)$ is 1-dimensional, $f_{u,v}$ must be a nonzero multiple of the homomorphism defined in [2].
Recall that there is a short exact sequence of $F\Sigma_n$-modules

\[ 0 \rightarrow S_{u-1}^\lambda \rightarrow S_u^\lambda \rightarrow S_u^\lambda \rightarrow 0. \]

Let $\theta_u^{-1}$ be a $F$-retraction of $\theta_u$. Then $f_{u,v} = \theta_u^{-1} \theta_{u,v} \theta_v$, by the previous paragraph.

Example 19. For $p = 5$ there is a one-box shift $f : S^{[4,2]} \rightarrow S^{[3,2,1]}$. Here $\lambda = [4,2,1]$ has removable residues $\alpha_1 = 3, \alpha_2 = 0$ and $\alpha_3 = 3$. Using the previous theorem, and Lemma 7 we compute

\[
\begin{array}{cccccccc}
& & & & & & 1 & 2 & 3 \\
& & & & & 1 & 2 & 3 & 4 \\
& & & & 1 & 2 & 3 & 4 & 5 \\
& & & 1 & 2 & 3 & 4 & 5 & 6 \\
& & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

Theorem 20. Suppose that $1 \leq v < u \leq m$ are such that $a_v = \alpha_u$, but none of $\alpha_v, \alpha_{v+1}, \ldots, \alpha_{u-1}$ equals $\alpha_u$. Then $D_u = D_{u-1}$.

Proof. The elements \( \{ e_t \in S_u^\lambda \setminus S_{u-1}^\lambda \} \) generate $S_u^\lambda$ as $S_{u-1}^\lambda$-module. Moreover Lemma 7 implies that $e_t \prod_{i=v}^{u}(L_n - \alpha_i) \equiv \prod_{i=v}^{u}(\alpha_u - \alpha_i) e_t \pmod{S_{u-1}^\lambda}$, for each $e_t \in S_u^\lambda \setminus S_{u-1}^\lambda$. As $\prod_{i=v}^{u}(\alpha_u - \alpha_i) \neq 0_F$, it is enough to show that $e_t \prod_{i=v}^{u}(L_n - \alpha_i) \in J^\lambda = \ker(\theta)$.

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