Projective modules and involutions

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Abstract

Let $G$ be a finite group, and let $\Omega := \{ t \in G \mid t^2 = 1 \}$. Then $\Omega$ is a $G$-set under conjugation. Let $k$ be an algebraically closed field of characteristic 2. It is shown that each projective indecomposable summand of the $G$-permutation module $k\Omega$ is irreducible and self-dual, whence it belongs to a real 2-block of defect zero. This, together with the fact that each irreducible $kG$-module that belongs to a real 2-block of defect zero occurs with multiplicity 1 as a direct summand of $k\Omega$, establishes a bijection between the projective components of $k\Omega$ and the real 2-blocks of $G$ of defect zero.

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Let $G$ be a finite group, with identity element $e$, and let $\Omega := \{ t \in G \mid t^2 = e \}$. Then $\Omega$ is a $G$-set under conjugation. In this note we describe the projective components of the permutation module $k\Omega$, where $k$ is an algebraically closed field of characteristic 2. By a projective component we mean an indecomposable direct summand of $k\Omega$ that is also a direct summand of a free $kG$-module. We show that all such components are irreducible, self-dual and occur with multiplicity 1.

This gives an alternative proof of Remark (2) on p. 254 of [5], and strengthens Corollaries 3 through 7 of that paper. In addition, we can give the following quick proof of Proposition 8 in [5]:

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Corollary 1. Suppose that $H$ is a strongly embedded subgroup of $G$. Then $k_H \uparrow^G \cong k_G \oplus \bigoplus_{i=1}^s P_i$ where $s \geq 0$ and the $P_i$ are pairwise nonisomorphic self-dual projective irreducible $k_G$-modules.

Proof. That $H$ is strongly embedded means that $|H|$ is even and $|H \cap H^g|$ is odd, for each $g \in G \setminus H$. Let $t \in H$ be an involution. Then clearly $C_G(t) \leq H$. So $k_H \uparrow^G$ is isomorphic to a submodule of $(k_{C_G(t)}) \uparrow^G$. Mackey’s theorem implies that every component of $k_H \uparrow^G$, other than $k_G$, is a projective $k_G$-module. Being projective, these modules must be components of $(k_{C_G(t)}) \uparrow^G$. The result now follows from Theorem 8.

Consider the wreath product $G \wr \Sigma$ of $G$ with a cyclic group $\Sigma$ of order 2. Here $\Sigma$ is generated by an involution $\sigma$ and $G \wr \Sigma$ is isomorphic to the semidirect product of the base group $G \times G$ by $\Sigma$. The conjugation action of $\sigma$ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. The elements of $G \wr \Sigma$ will be written $(g_1, g_2), (g_1, g_2) \sigma$ or $\sigma$.

We shall exploit the fact that $kG$ is an $k(G \times G)$-module via $x \cdot (g_1, g_2) := g_1^{-1}xg_2$, for each $x \in kG$, and $g_1, g_2 \in G$. The action of $\Sigma$ on $kG$ is induced by the permutation action of $\sigma$ on the distinguished basis $G$ of $kG$: $g^\sigma := g^{-1}$, for each $g \in G$. Clearly $\sigma$ acts as an involutory $k$-algebra anti-automorphism of $kG$. If it follows that the actions of $G \times G$ and $\Sigma$ on $kG$ are compatible with the group relations in $G \wr \Sigma$.

By a block of $kG$, or a 2-block of $G$, we mean an indecomposable $k$-algebra direct summand of $kG$. Each block has associated to it a primitive idempotent in $Z(kG)$, a Brauer equivalence class of characters of irreducible $kG$-modules and a Brauer equivalence class, modulo 2, of ordinary irreducible characters of $G$. A block has defect zero if it is a simple $k$-algebra, and is real if it contains the complex conjugates of its ordinary irreducible characters. Theorem 8 establishes a bijection between the real 2-blocks of $G$ that have defect zero and the projective components of $kG$.

We could equally well work over a complete discrete valuation ring $R$ of characteristic 0, whose field of fractions $F$ is algebraically closed, and whose residue field $R/J(R)$ is $k$. So we use $O$ to indicate either of the commutative rings $k$ or $R$.

All our modules are right-modules. We denote the trivial $OG$-module by $O_G$. If $M$ is an $OG$-module, we use $M \downarrow_H$ to denote the restriction of $M$ to $H$. If $H$ is a subgroup of $G$ and $N$ is an $OH$-module, we use $N \uparrow^G_H$ to denote the induction of $N$ to $G$. Whenever $g \in G$, we write $g$ for $(g, g) \in G \times G$, and we set $X \uparrow := \{x \mid x \in X\}$, for each $X \subset G$. Other notation and concepts can be found in a standard textbook on modular representation theory, such as [1] or [4].

If $B$ is a block of $OG$, then so too is $B^\sigma = \{x^\sigma \mid x \in B\}$. We call $B$ a real block if $B = B^\sigma$. Our first result describes the components of $OG$ as $OG \wr \Sigma$-module:

Theorem 2. There is an indecomposable decomposition of $OG$ as $OG \wr \Sigma$-module:

$$OG = B_1 \oplus \cdots \oplus B_r \oplus (B_{r+1} + B_{r+1}^\sigma) \oplus \cdots \oplus (B_{r+s} + B_{r+s}^\sigma).$$

Here $B_1, \ldots, B_r$ are the real 2-blocks and $B_{r+1}, B_{r+1}^\sigma, \ldots, B_{r+s}, B_{r+s}^\sigma$ are the nonreal 2-blocks of $G$. 

Proof. By a suitable permutation of the basis vectors of a $kG \wr \Sigma$-module, we can provide a basis of $OG$ that realises $OG \wr \Sigma$ as the wreath product of $kG \wr \Sigma$ by an appropriate $\Sigma$-module. The theorem now follows from the well-known fact that $kG \wr \Sigma$ is isomorphic to the semidirect product of the base $G \times G$ by $\Sigma$.

Theorem 3. Suppose that $G$ is a group and $\sigma$ is an involution of $G$.

Let $\Sigma = \{1, \sigma\}$, $G \wr \Sigma := G \times G$, and $A = kG \wr \Sigma$. Then $A$ is an $OG \wr \Sigma$-module, and $OG \wr \Sigma$ is a $kG \wr \Sigma$-module.

If $B$ is a $kG \wr \Sigma$-module, then $B^\sigma$ is a $kG \wr \Sigma$-module too. We say that $B$ is an $\sigma$-real $kG \wr \Sigma$-module if $B = B^\sigma$. The real $\sigma$-blocks are the indecomposable $\sigma$-real $kG \wr \Sigma$-modules. The nonreal $\sigma$-blocks are the indecomposable $\sigma$-nonreal $kG \wr \Sigma$-modules.

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Proof. This follows from the well-known indecomposable decomposition of $OG$, as an $O(G \times G)$-module, into a direct sum of its blocks, and the fact that $B_i^\sigma = B_i$ for $i = 1, \ldots, r$, and $B_{r+j}^\sigma = B_{r+j}$ for $j = 1, \ldots, s$. □

An obvious but useful fact is that $OG$ is a permutation module:

**Lemma 3.** The $OG \wr \Sigma$-module $OG$ is isomorphic to the permutation module $(OG \times \Sigma)^{\wr G \wr \Sigma}$.

**Proof.** The elements of $G$ form a $G \wr \Sigma$-invariant basis of $OG$. Moreover if $g_1, g_2 \in G$, then $g_2 = g_1 (g_1, g_2)$. So $G$ is a transitive $G \wr \Sigma$-set. The stabilizer of $g \in OG$ in $G \wr \Sigma$ is $G \times \Sigma$. The lemma follows from these facts. □

Let $C$ be a conjugacy class of $G$. Set $C^o := \{ c \in G \mid c^{-1} \in C \}$. Then $C^o$ is also a conjugacy class of $G$, and $C \cup C^o$ can be regarded as an orbit of $G \times \Sigma$ on the $G \wr \Sigma$-set $G$. As such, the corresponding permutation module $O(C \cup C^o)$ is a $OG \times \Sigma$-direct summand of $OG$. If $C = C^o$, we call $C$ a real class of $G$. In this case for each $c \in C$ there exists $x \in G$ such that $c^x = c^{-1}$. The point stabilizer of $c$ in $G \times \Sigma$ is $CG(c)(\langle x \rangle \Sigma)$. So

$$OC \cong (OC_{G(c)}(\langle x \rangle \Sigma))^{\wr G \times \Sigma}. $$

If $C \neq C^o$, we call $C$ a nonreal class of $G$. In this case the point stabilizer of $c \in C \cup C^o$ in $G \times \Sigma$ is $CG(c)$. So

$$OC(C \cup C^o) \cong (OC_{G(c)}(\langle x \rangle \Sigma))^{\wr G \times \Sigma}. $$

Suppose now that the real classes are $C_1, \ldots, C_t$ and that the nonreal classes are $C_{t+1}, C_{t+1}^o, \ldots, C_{t+r}, C_{t+r}^o$. Then we have:

**Lemma 4.** There is a decomposition of $OG$ as an $OG \times \Sigma$-permutation module:

$$OG = OC_1 \oplus \cdots \oplus OC_t \oplus OC_{t+1} \oplus C_{t+1}^o \oplus \cdots \oplus OC_{t+r} \oplus C_{t+r}^o \oplus OC_{t+r+1} \oplus C_{t+r+1}^o.$$

**Proof.** This follows from Lemma 3 and the discussion above. □

By a quasi-permutation module we mean a direct summand of a permutation module. Our next result is Lemma 9.7 of [1]. We include a proof for the convenience of the reader.

**Lemma 5.** Let $M$ be an indecomposable quasi-permutation $OG$-module and suppose that $H$ is a subgroup of $G$ such that $M \downarrow H$ is indecomposable. Then there is a vertex $V$ of $M$ such that $V \cap H$ is a vertex of $M \downarrow H$. If $H$ is a normal subgroup of $G$, then this is true for all vertices of $M$.

**Proof.** Let $U$ be a vertex of $M$. As $O_{U \mid M \downarrow U}$ we have $O_{U \cap H \mid (M \downarrow H) \downarrow U \cap H}$. But $U \cap H$ is a vertex of $O_{U \cap H}$. So Mackey’s theorem implies that there exists a vertex $W$ of $M \downarrow H$ such that $U \cap H \subseteq W$. 


As $M \downarrow H$ is a component of the restriction of $M$ to $H$, Mackey’s theorem shows that there exists $g \in G$ such that $W \subseteq U^g \cap H$. Now $U^g$ is a vertex of $M$. So by the previous paragraph, and the uniqueness of vertices of $M \downarrow H$ up to $H$-conjugacy, there exists $h \in H$ such that $U^g \cap H = W^h$. Comparing cardinalities, we see that $W = U^g \cap H$. So $U^g \cap H$ is a vertex of $M \downarrow H$.

Suppose that $H$ is a normal subgroup of $G$. Then $U \cap H \leq W$ and $W = U^g \cap H = (U \cap H)^g$ imply that $U \cap H = W$. □

R. Brauer showed how to associate to each block of $OG$ a $G$-conjugacy class of 2-subgroups, its so-called defect groups. It is known that a block has defect zero if and only if its defect groups are all trivial. J.A. Green showed how to associate to each indecomposable $OG$-module a $G$-conjugacy class of 2-subgroups, its so-called vertices. He also showed how to identify the defect groups of a block using its vertices as an indecomposable $OG(\times G)$-module.

**Corollary 6.** Let $B$ be a block of $OG$ and let $D$ be a defect group of $B$. If $B$ is not real then $D$ is a vertex of $B + B^o$, as $OG \times \Sigma$-module. If $B$ is real, then there exists $x \in N_G(D)$, with $x^2 \in D$, such that $D(x\sigma)$ is a vertex of $B$, as $OG \times \Sigma$-module. In particular, $\Sigma$ is a vertex of $B + B^o$ if and only if $B$ is a real 2-block of $G$ that has defect zero.

**Proof.** J.A. Green showed in [2] that $D$ is a vertex of $B$, when $B$ is regarded as an indecomposable $OG(\times G)$-module. Suppose first that $B$ is not real. Then $B + B^o = (B \downarrow G \times G)(G)_{G \times \Sigma}$, for instance by Corollary 8.3 of [1]. It follows that $B + B^o$ has vertex $D$, as an indecomposable $OG \times \Sigma$-module.

Suppose then that $B = B + B^o$ is real. Lemma 3 shows that $B$ is $G \times \Sigma$-projective. So we may choose a vertex $V$ of $B$ such that $V \subseteq G \times \Sigma$. Moreover, $B$ is a quasi-permutation $OG \times \Sigma$-module, and its restriction to the normal subgroup $G \times G$ is indecomposable. Lemma 5 then implies that $V \cap (G \times G) = V \cap G$ is a vertex of $B \downarrow G \times G$. So by Green’s result, we may choose $D$ so that $V \cap G = D$. Now $G \times G$ has index 2 in $G \times \Sigma$. So Green’s indecomposability theorem, and the fact that $B \downarrow G \times G$ is indecomposable, implies that $V \not\subseteq (G \times G)$. It follows that there exists $x \in N_G(D)$, with $x^2 \in D$, such that $V = D(x\sigma)$.

If $B$ has defect zero, then $D = (e)$. So $x^2 = e$. In this case, $(x\sigma) = \Sigma(x^2)$ is $G \times \Sigma$-conjugate to $\Sigma$. So $\Sigma$ is a vertex of $B$. Conversely, suppose that $\Sigma$ is a vertex of $B + B^o$. The first paragraph shows that $B$ is a real block of $G$. Moreover $B$ has defect zero, as $\Sigma \cap G = (e)$. □

We quote the following result of Burry, Carlson and Puig [4, 4.4.6] on the Green correspondence:

**Lemma 7.** Let $V \leq H \leq G$ be such that $V$ is a $p$-group and $N_G(V) \leq H$. Let $f$ denote the Green correspondence with respect to $(G, V, H)$. Suppose that $M$ is an indecomposable $OG$-module such that $M \downarrow H$ has a component $N$ with vertex $V$. Then $V$ is a vertex of $M$ and $N = f(M)$. 
We can now prove our main result. Part (ii) is Remark (2) on p. 254 of [5], but our proof is independent of the proof given there.

**Theorem 8.**

(i) Let \( t \in G \), with \( t^2 = e \). Suppose that \( P \) is an indecomposable projective direct summand of \( (\mathcal{O}_G(1))^+ \). Then \( P \) is irreducible and self-dual and occurs with multiplicity 1 as a component of \( (\mathcal{O}_G(1))^+ \). In particular \( P \) belongs to a real 2-block of \( G \) that has defect zero.

(ii) Suppose that \( M \) is a projective indecomposable \( \mathcal{O}_G \)-module that belongs to a real 2-block of \( G \) that has defect zero. Then there exists \( s \in G \), with \( s^2 = e \), such that \( M \) is a component of \( (\mathcal{O}_G(1))^+ \). Moreover, \( s \) is uniquely determined up to conjugacy in \( G \).

**Proof.** If \( t = e \) then \( P = \mathcal{O}_G \). So \( P \) is irreducible and self-dual. The assumption that \( P \) is projective and the fact that \( \dim_{\mathcal{O}_G}(P) = 1 \) implies that \( |G| \) is odd. So all blocks of \( \mathcal{O}_G \), in particular the one containing \( P \), have defect zero.

Now suppose that \( t \neq e \). Let \( T \) be the conjugacy class of \( G \) that contains \( t \). The permutation module \( \mathcal{O}T \) is a direct summand of the restriction of \( \mathcal{O}_G \) to \( G \times T \). Regard \( P \) as an \( \mathcal{O}_G \)-module. Let \( I(P) \) be the inflation of this module to \( G \times T \). Then \( I(P) \) is a component of \( \mathcal{O}T \). As \( T \) is contained in the kernel of \( I(P) \), and \( P \) is a projective \( \mathcal{O}_G \)-module, it follows that \( I(P) \) has vertex \( T \) as an indecomposable \( \mathcal{O}_G \times T \)-module.

By Lemma 2, and the Krull–Schmidt theorem, there exists a 2-block \( B \) of \( G \) such that \( I(P) \) is a component of the restriction \( (B + B^\circ)^\uparrow_{G \times T} \). An easy computation shows that \( N_{G/T}(T) = G \times T \). It then follows from Lemma 7 that \( (B + B^\circ) \) has vertex \( T \) and also that \( I(P) \) is the Green correspondent of \( (B + B^\circ) \) with respect to \( (G : T, \Sigma, G \times T) \). We conclude from Corollary 6 that \( B \) is a real 2-block of \( G \) that has defect zero.

Let \( \hat{B} \) be the 2-block of \( G : T \) that contains \( B \). Then \( \hat{B} \) is real and has defect group \( T \). Let \( \hat{A} \) be the Brauer correspondent of \( \hat{B} \). Then \( \hat{A} \) is a real 2-block of \( G \times T \) that has defect group \( T \). Now \( \hat{A} = \hat{A} \otimes \Omega \Sigma \), where \( \hat{A} \) is a real 2-block of \( \mathcal{O}_G \) that has defect zero. In particular \( \hat{A} \) has a unique indecomposable module, and this module is projective, irreducible and self-dual. Corollary 14.4 of [1] implies that \( I(P) \) belongs to \( \hat{A} \). So \( P \) belongs to \( \hat{A} \). We conclude that \( P \) is irreducible and self-dual and belongs to a real 2-block of \( G \) that has defect zero.

Now \( B \) occurs with multiplicity 1 as a component of \( \mathcal{O}_G \), and \( I(P) \) is the Green correspondent of \( B \) with respect to \( (G : T, \Sigma, G \times T) \). So \( I(P) \) has multiplicity 1 as a component of the restriction \( \mathcal{O}_G(1) \) to \( G \times T \). It follows that \( P \) occurs with multiplicity 1 as a component of \( (\mathcal{O}_G(1))^+ \), and with multiplicity 0 as a component of \( (\mathcal{O}_G(1))^+ \), for \( r \in G \) with \( r^2 = e \), but not \( G \)-conjugate to \( t \). This completes the proof of part (i).

Let \( R \) be a real 2-block of \( G \) that has defect zero. Then \( R \) has vertex \( T \) as an indecomposable \( \mathcal{O}_G \times T \)-module. So its Green correspondent \( f(R) \), with respect to \( (G : T, \Sigma, G \times T) \), is a component of the restriction of \( \mathcal{O}_G \) to \( G \times T \) that has vertex \( T \). Lemma 4 and the Krull–Schmidt theorem imply that \( f(R) \) is isomorphic to a component of \( \mathcal{O}(C \cup C^\circ) \), for some conjugacy class \( C \) of \( G \). Now \( \Sigma \) is a central subgroup of \( G \times T \). So \( \Sigma \) must be a subgroup of the point stabilizer of \( C \cup C^\circ \) in \( G \times T \). It follows that \( s^2 = e \), for each \( s \in C \).
Let $N$ denote the restriction of $f(R)$ to $\overline{G}$, and consider $N$ as an $OG$-module. We have just shown that $N$ is a component of $(OC_{G}(s))^{G}$. Arguing as before, we see that $N$ is an indecomposable projective $OG$-module that belongs to a real 2-block of $G$ that has defect zero.

The last paragraph establishes an injective map between the real 2-blocks of $G$ that have defect zero and certain projective components of $O\Omega$. As each block of defect zero contains a single irreducible $OG$-module, this map must be onto. It follows that the module $M$ in the statement of the theorem is a component of some permutation module $(OC_{G}(s))^{G}$, where $s \in G$ and $s^2 = e$. The fact that $s$ is determined up to $G$-conjugacy now follows from the last statement of the proof of part (i). This completes the proof of part (ii). \[ \square \]

It is possible to simplify the above proof by showing that if $B$ is a real 2-block of $G$ that has defect zero, then its Green correspondent, with respect to $(G : \Sigma, \Sigma, G \times \Sigma)$ is $M^{R}$, where $M^{R}$ is the Frobenius conjugate of the unique irreducible $OG$-module that belongs to $B$.

Suppose that $R$ is a complete discrete valuation ring and that $L$ is an $RC_{G}(t)$-module, where $L$ has $R$-rank 1 and $O^{2}(C_{G}(t))$ acts trivially on $L$. Then the 2-modular reduction of $L$ is the trivial $kC_{G}(t)$-module, although $L$ is not necessarily the trivial $RC_{G}(t)$-module. Now each projective irreducible $kG$-module lifts to a projective irreducible $RG$-module. So the conclusions of part (i) of the above theorem apply to $L^{G}$; all of its projective components are irreducible and self-dual. We thank the referee for pointing out this extension of our result.

The proof of Theorem 8 hints at the fact that we have some 2-local control over all the components of $(OC_{G}(t))^{G}$. The investigation of special properties of such components is continued in [3].

**Corollary 9.** Let $\Omega = \{t \in G \mid t^2 = e\}$. Then there is a bijection between the real 2-blocks of $G$ that have defect zero and the projective components of $O\Omega$.

Here is a sample application. It was suggested to me by G.R. Robinson.

**Corollary 10.** Let $n \geq 1$ and let $t$ be an involution in the symmetric group $\Sigma_n$. If $n = m(m+1)/2$ is a triangular number, and $t$ is a product of $\lfloor (m^2 + 1)/4 \rfloor$ commuting transpositions, then there is a single projective irreducible $O\Sigma_n$-module, and this module is the unique projective component of $(OC_{\Sigma_n}(t))^{\Sigma_n}$. For all other values of $n$ or nonconjugate involutions $t$, the modules $(OC_{\Sigma_n}(t))^{\Sigma_n}$ are projective free.

**Proof.** We give a proof of the following result in [3, Corollary 8.4]: Let $G$ be a finite group, let $B$ be a real 2-block of $G$ of defect zero, and let $\chi$ be the unique irreducible character in $B$. Then there exists a 2-regular conjugacy class $C$ of $G$ such that $C = C^0$, $|C_G(c)|$ is odd, for $c \in C$, and $\chi(c)$ is nonzero, modulo a prime ideal containing 2. Moreover, there exists an involution $t \in G$ such that $c^t = c^{-1}$, and for this $t$ we have $(\chi_{C_G(t)}, 1_{C_G(t)}) = 1$.

The existence of $t$ was shown in [5]. The identification of $t$ using the class $C$ was first shown by R. Gow (in unpublished work).
Suppose that \((O_{\Sigma_n(t)} \uparrow \Sigma_n)\) has a projective component. Then \(\Sigma_n\) has a 2-block of defect zero, by Theorem 8. The 2-blocks of \(\Sigma_n\) are indexed by triangular partitions \(\mu = \begin{bmatrix} m, m-1, \ldots, 2, 1 \end{bmatrix}\), where \(m\) ranges over those natural numbers for which \(n - m(m+1)/2\) is even. Moreover, the 2-block corresponding to \(\mu\) has defect zero if and only if \(n = m(m+1)/2\). In particular, we can assume that \(n = m(m+1)/2\), for some \(m \geq 1\).

Let \(B\) be the unique 2-block of \(\Sigma_n\) that has defect zero, let \(\chi\) be the unique irreducible character in \(B\) and let \(g \in \Sigma_n\) have cycle type \(\lambda = \begin{bmatrix} 2m-1, 2m-5, \ldots \end{bmatrix}\). Then \(|C_{\Sigma_n}(g)|\) is odd. As the parts of \(\lambda\) are the “diagonal hooklengths” of \(\mu\), the Murnaghan–Nakayama formula shows that \(\chi(g) = 1\). Now \(\lambda\) has \([(m-1)/2]\) nonzero parts. So \(g\) is inverted by an involution \(t\) that is a product of \((n - [(m-1)/2])\)/2 = \([m^2 + 1]/4\) commuting transpositions. It follows from Theorem 8 and the previous paragraph that the unique irreducible projective \(B\)-module occurs with multiplicity 1 as a component of \((O_{\Sigma_n(t)} \uparrow \Sigma_n)\). The last statement of the corollary now follows from Theorem 8. □

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