STRONG DOUBLING CONDITIONS

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Abstract. We show that the class of strong doubling measures depends essentially on the parameter $t$, and that the measure of the boundary layer of a QHBC domain decays geometrically, if the measure is suitably strong doubling.

0. Introduction

Various areas of analysis utilize doubling measures, i.e. positive Borel measures on $\mathbb{R}^n$ satisfying (some variation of) the condition:

$$\mu(B(x,r)) \leq C \mu(B(x,r/2)), \quad \text{for all } x \in \mathbb{R}^n, \ r > 0. \quad (0.1)$$

For instance, Chapter I of [S2] investigates many questions in harmonic analysis within a general framework involving a measure that satisfies a doubling condition relative to a set of generalized balls in $\mathbb{R}^n$, and [HKM] develops the potential theory of a certain class of degenerate elliptic partial differential equations that involve admissible weights, where a weight $w$ is admissible if the measure $w\, dx$ satisfies certain conditions including (0.1).

Much of this analysis takes place on an open subset $\Omega$ of $\mathbb{R}^n$, rather than on all of $\mathbb{R}^n$ (for instance, this is often the case for PDE-related analysis). Some such results require only a local doubling condition for balls $B(x,2r) \subset \Omega$, for instance, but often a stronger form of doubling is required. It is then quite common to assume that the measure is defined on all of $\mathbb{R}^n$ and satisfies (0.1); this, for example, is the approach adopted in [HKM] for the definition of an admissible weight. However, there exist rather nice measures defined on an open set $\Omega$ which are not restrictions of global doubling measures, e.g. power-weight measures $d\mu = \delta^a_{\Omega} \, dx$ for certain domains $\Omega$, where $\delta_{\Omega}(x)$ is the distance from $x$ to $\partial \Omega$, and $a > 0$. The author wishes to thank Paul MacManus for kindly providing an explicit example of this type (given at the end of Section 1).

One doubling condition applicable to measures on $\Omega$ is the boundary doubling condition:

$$\mu(B(x,r) \cap \Omega) \leq C \mu(B(x,r/2) \cap \Omega), \quad \text{for all } x \in \Omega, \ r > 0. \quad (0.2)$$

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This condition, however, places restrictions on $\Omega$ as well as on $\mu$, since even Lebesgue measure does not always satisfy (0.2) (see the proof of Theorem 1.1). The concept of a strong doubling measure, employed in [BKL] and [BO] to prove inequalities of Poincaré and Trudinger type, is an attractive intermediate option; there are actually a family of such strong doubling conditions indexed by a parameter $1 < t < \infty$ (see Section 1). These conditions are strong enough to do some non-local analysis, but weaker than boundary doubling. Additionally, they are all satisfied by the measure $\delta^a \, dx$, $a \geq 0$, no matter how bad the geometry of the domain $\Omega$.

In Section 1, we determine how strong doubling conditions relate to each other and to other doubling conditions; in particular, we show that all strong doubling conditions are different, since there exist measures which are strong doubling for all parameters less than $t$, but not for parameter $t$. In Section 2, we prove that if a measure is appropriately strong doubling on a QHBC domain $\Omega$, then the measure of the part of $\Omega$ lying within a distance $\epsilon$ of $\partial \Omega$ is dominated by a power of $\epsilon$. This result, which generalizes a result of Smith and Stegenga on the Minkowski dimension of $\partial \Omega$, has been used in [BO, Theorem 3.10] to prove a theorem on Trudinger-type inequalities.

1. Various doubling conditions

Throughout this paper, $\Omega$ is a proper open subset of $\mathbb{R}^n$, which we may further restrict as necessary. If $B = B(x,r)$ is a ball, and $t > 0$, we write $tB = B(x, tr)$ (and so $t^{-1}B = B(x, r/t)$). We also write $\delta_{\Omega}(x) \equiv \text{dist}(x, \partial \Omega)$, $x \in \Omega$, and define the quasi hyperbolic length of a rectifiable path $\gamma \subset \Omega$ to be

$$k_{\Omega}(\gamma) = \int_{\gamma} \delta_{\Omega}(x)^{-1} ds.$$ 

The quasi hyperbolic distance between $x, y \in \Omega$, $k_{\Omega}(x, y)$, is then defined to be the infimum of $k_{\Omega}(\gamma)$, as $\gamma$ ranges over all paths linking $x$ and $y$. There exists a quasi hyperbolic geodesic between any pair of points $x, y \in \Omega$, i.e. a path $\gamma_{x,y}$ such that $k_{\Omega}(x, y) = k_{\Omega}(\gamma_{x,y})$; see [GO].

A (necessarily bounded) domain $\Omega$ satisfies a quasi hyperbolic boundary condition (more briefly, $\Omega$ is QHBC) with respect to its QHBC center $x_0 \in \Omega$ if there exists a constant $C \geq 1$ such that for all $x \in \Omega$,

$$k_{\Omega}(x, x_0) \leq C \log \left( \frac{C}{\delta(x)} \right).$$

The QHBC path for $x$ is the quasi hyperbolic geodesic for $x, x_0$, and the QHBC constant of $\Omega$, denoted $C_{\Omega}$, is the smallest value of $C$ for which the above inequality is valid.

We say that a bounded domain $\Omega$ is a John domain with respect to its John center $x_0 \in \Omega$ if there exists a constant $K \geq 1$ such that for all $x \in \Omega$, there is a path $\gamma = \gamma_x : [0,l] \to \Omega$ parametrized by arclength satisfying $\gamma(0) = x$, $\gamma(l) = x_0$, 


and $\delta(\gamma(t)) \geq t/K$. We call $\gamma_x$ the John path for $x$, and we define $K_\Omega$, the John constant of $\Omega$, to be the smallest value of $K$ for which the above inequality is valid.

Clearly every John domain is a QHBC domain, but it is not difficult to construct examples of non-John QHBC domains (e.g. see [BO, Section 5]). Note that the choice of center point $x_0 \in \Omega$ in the definitions of John and QHBC domains is unimportant, in the sense that if $\Omega$ is John (or QHBC) with respect to one point, it is John (or QHBC) with respect to all of its points (of course, the John/QHBC constant tends to infinity as we let $x_0$ approach $\partial \Omega$).

Suppose that $0 < t \leq \infty$ and that $\mu$ is a positive Borel measure on $\Omega$. We say that $\mu$ is $t$-doubling on $\Omega$, denoted $\mu \in D_t(\Omega)$, if there exists a constant $C$ such that

$$\mu(B \cap \Omega) \leq C \mu(2^{-1}B \cap \Omega) < \infty$$

whenever $B$ is a ball for which $t^{-1}B \subset \Omega$ (in the case $t = \infty$, we merely require the center of $B$ to lie in $\Omega$, or equivalently in $\Omega$). We denote by $C_{\mu,t}$ the smallest such constant $C$ for which this doubling condition is true ($0 < t \leq \infty$).

Note that the $t$-doubling condition imposes restrictions on the boundary behaviour of the measure precisely when $t \geq 1$. We say that a $t$-doubling measure $\mu$ is a locally doubling if $t < 1$, strong doubling if $t > 1$, and boundary doubling if $t = \infty$. Obviously, strong doubling is logically stronger than local doubling but weaker than boundary doubling. In fact, it is not difficult to construct examples of a measure that is local doubling but not strong doubling, or strong doubling but not boundary doubling. Whether or not strong doubling depends on the parameter $t \in (1,\infty)$ is a more difficult question which we now answer.

**Theorem 1.1.** Suppose $0 < t < t' \leq \infty$. If $t' \geq 1$, $D_{t'}(\Omega) \setminus D_t(\Omega)$ is non-empty for some QHBC domain $\Omega \subset \mathbb{R}^n$. If $t' < 1$, then $D_t(\Omega) = D_{t'}(\Omega)$ for every proper open set $\Omega$.

Before proving this theorem, we first state a simple but useful lemma.

**Lemma 1.2.** A sphere $S \subset \mathbb{R}^n$ of radius $a > 0$ can be covered by balls $\{B_i\}_{i=1}^m$, centered on $S$ and of radius $b \in (0,a)$, for some $m$ dependent only on $n$ and $b/a$.

**Proof.** We choose a sequence of disjoint balls $B'_1, B'_2, \ldots$, centered on $S$ and of radius $b/3$ as long as we can continue to do so; this process must halt in a bounded number of steps since each ball covers a fixed fraction (dependent only on $b/a, n$) of the surface measure of $S$. If the resulting balls are $B'_1, \ldots, B'_m$, then the required balls are $B_i = 3B'_i$, $1 \leq i \leq m$. ☐

**Proof of Theorem 1.1.** The equivalence of all local doubling conditions is intuitively rather obvious, but we prove it for completeness. Assume that $\mu \in D_t(\Omega)$ for some $t < 1$, and so $\mu(B) \leq C \mu(2^{-1}B)$ whenever $B = B(x,r)$, $0 < r \leq t\delta_\Omega(x)$. We fix such a ball $B(x,r)$ with $r = t\delta_\Omega$, and write $c = (2-2t)/(2-t)$. Applying Lemma 1.2 with $a = (1-c/2)r$, $b = cr/4$, to the sphere $S = \{y : |x-y| = a\}$, we get balls
$B_1, \ldots, B_m$ covering $S$, where $m$ is bounded by some number dependent only on $n$ and $t$. Our choice of parameters ensures that

$$2B_i \subset B(x, r),$$
$$4B_i \supset B(x, (1 + (c/4))r) \setminus B(x, r),$$
$$4t^{-1}B_i \subset \Omega.$$  

We deduce that $\mu \in D_{f(t)}$, where $f(t) = (5t - 3t^2)/(4 - 2t) = (1 + c/4)t$. Defining $t_0 = t$ and $t_k = f(t_{k+1})$ for all $k > 0$, it follows iteratively that $\mu \in D_{t_k}$ for every $k > 0$. Note that $ct/4 < 1 - t$ and so the sequence $(t_k)$ is increasing and bounded by 1. Since $f$ is continuous on $(0, 2)$ and 1 is the only fixed point there, we deduce that $t_k$ tends to 1 as $k \to \infty$. Thus $\mu \in D_{t'}$ for all $t' < 1$, as required.

Letting $\Omega$ be the unit ball in $\mathbb{R}^n$, it is easy to find $\mu \in D_t(\Omega) \setminus D_1(\Omega)$ whenever $t < 1$. For example, $d\mu = (1 - |x|)^{-1} dx$ is such a measure. Alternatively, we could take $d\mu = [\log(2/(1 - |x|))]^{-2}(1 - |x|)^{-1} dx$; in this latter case, $\mu(\Omega) < \infty$.

In the remaining cases, we give only planar counterexamples to equality; these are easily modified to give counterexamples in any larger dimension. The domains we use will consist of a central square with small narrow pieces attached. It is convenient for us to take as our central square

$$Q_0 = \{(x, y) \in \mathbb{R}^2 : -1 < x < 0, 0 < y < 1\}.$$  

We first consider the case $t' = \infty$ (even though it follows from the case $t' < \infty$), because we can produce a counterexample here with $\mu$ equal to Lebesgue measure. Note first that Lebesgue measure lies in $D_t(\Omega)$ for all $t < \infty$, regardless of the domain $\Omega$. We define $\Omega$ to be the union of $Q_0$ and the rectangles

$$R_k = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 2^{-k}, 1 - 2^{-k}(1 + 1/k) < y < 1 - 2^{-k}\}.$$  

Then $\cdot$ is not boundary doubling because $|2^{-1}B_k \cap \Omega| \approx |B_k \cap \Omega|/k$, where $B_k$ is the ball whose center is the same as the center of $R_k$ and whose radius is $2^{-k}$. The only possible obstacle to $\Omega$ being QHBC is the narrowness of the rectangles $R_k$. Since the length-to-width ratio of $R_k$, i.e. $k$, is dominated by the logarithm of the reciprocal of $R_k$’s diameter, this is not a genuine obstacle, and it is easy to check that $\Omega$ is QHBC.

To prove the remaining cases, it suffices to find, for all $1 < t < t' < \infty$, a QHBC domain $\Omega$ and a measure $\mu$ such that $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. By elementary geometry, we note that if $B$ is a ball inscribed in the cone $K_a = \{ (x, y) : |y| < ax \}$, $a > 0$, then the dilate $sB$, $s > 1$, contains the vertex of $K_a$ if and only if $a > f(s) = [s^2 - 1]^{-1/2}$. We shall define $\Omega$ to be a union of $Q_0$ and a sequence of diamond-shaped sets $S_k$. First, we define the preliminary diamond-shaped sets

$$S_k' = \{(x, y) \in \mathbb{R}^2 : 0 \leq |x| < x_k, |y| < f(t')|x - x_k|\},$$

where $x_k = 2^{-k}c$, and $c = \min\{1, |4f(t')|^{-1}\}$. We then write $y_k = 1 - 2^{-k}$ and define $S_k$ to be the translate of $S_k'$ by the vector $(x_k - x_k^2, y_k)$. Note that the
sets $S_k$ have a small overlap with $Q_0$ but are disjoint from each other. The sets $S_k$ are of a fixed length-to-width ratio, but there is a new potential obstacle to $\Omega$ being a QHBC domain: each $S_k$ is attached to $Q_0$ by a narrow neck whose width is proportional to $x_k^2$. However, it is a routine exercise to check that if “satellite pieces” (such as $S_k$) are adjoined to the main part of the domain via bottlenecks of width proportional to a fixed power of the length of the satellite, then this does not destroy the QHBC condition. Consequently, $\Omega$ is QHBC.

Let us denote by $U_k$ and $V_k$ the vertices $(-x_k^2, y_k)$ and $(2x_k - x_k^2, y_k)$, respectively, of $S_k$. Defining

$$g_k(x) = x - |x - x_k + x_k^2|, \quad x \in \mathbb{R},$$
$$w_s(x, y) = \frac{|x_k^2/y_k(x)|^s}{(2 - s)x_k}, \quad (x, y) \in S_k,$$
$$d\mu_s = w_s(x, y) \, dx \, dy, \quad (x, y) \in S_k,$$

we see that $\mu_s(S_k) < \infty$ for $0 < s < 2$, but not for $s = 2$. Furthermore, as $s \to 2^-$, more and more of the $\mu_s$-mass of $S_k$ is concentrated closer and closer to $U_k$ and $V_k$. More precisely,

$$\lim_{s \to 2^-} \frac{\mu_s(\{X \in S_k : \min(|X - U_k|, |X - V_k|) < (2 - s)x_k\})}{\mu_s(S_k)} = 1.$$

By a routine calculation, this last limit reduces to the fact that $\lim_{t \to 0^+} t^t = 1$.

We are now ready to define a measure $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. Specifically, we take $d\mu \equiv w(x, y) \, dx \, dy$, where

$$w(x, y) = \begin{cases} 1, & (x, y) \in Q_0, \\ \frac{|x_k^2/y_k(x)|^{2-2/k}}{(2 - s)x_k}, & (x, y) \in S_k \setminus Q_0. \end{cases}$$

Note that $w$ is continuous across the necks of the sets $S_k$ (i.e. at $x = 0$) and, by the above considerations, most of the $\mu$-measure of $S_k$ is concentrated very near $V_k$ if $k$ is large. Considering balls inscribed in $S_k$ near this vertex, we deduce that any $t'$-doubling condition is violated for sufficiently large $k$. By contrast, $\mu$ is $t$-doubling for all $0 < t < t'$. To see this, note that balls centered in $S_k$ satisfy a $t$-doubling condition (because their $t^{-1}$-dilates stay away from $V_k$), and that balls centered in $Q_0$ actually satisfy an $\infty$-doubling condition (because the average value of $w$ on $S_k$ is bounded, as can easily be checked). \qed

In the above proof, we chose $\Omega$ to be the unit ball when defining a locally doubling measure on $\Omega$ which is not 1-doubling. By contrast, the fact that, for $1 < t \leq \infty$, the $D_t$-conditions are all distinct, made use of a domain which, although QHBC, was nevertheless rather nasty. We now show that such nastiness is in fact unavoidable.
Proposition 1.3. If $\Omega$ is a John domain, then $D_t(\Omega) = D_\infty(\Omega)$ for all $t \geq t_0$, where $t_0$ depends only on $K_\Omega$, the John constant of $\Omega$.

Proof. Let $x_0$ be the John center of $\Omega$. We fix a ball $B = B(x, r), x \in \Omega$. If $x_0 \in B$, then either $B \subset \Omega$, or $B$ contains a ball of radius $\delta_\Omega(x_0)/2$. In both cases, the required estimate

$$
\mu(B \cap \Omega) \leq C \mu(2^{-1}B \cap \Omega) < \infty
$$

follows easily from the assumption that $\mu \in D_t(\Omega)$ for sufficiently large $t = t(K_\Omega)$. Thus we may assume that $x_0 \notin B$. We choose any point $y$ on the John path for $x$ with respect to $x_0$ which lies in the annulus $B(x, r/3) \setminus B(x, r/6)$. The John condition ensures that $B' = B(y, r') \subset \Omega$ where $r' = r/6K_\Omega$. Since $B' \subset 2^{-1}B$ and $8K_\Omega B' \supset B$, it follows that $\mu(B \cap \Omega) \leq C \mu(2^{-1}B \cap \Omega)$ if $\mu \in D_{8K_\Omega}(\Omega)$. \qed

We end this section by giving an example, essentially due to Paul MacManus, of a strong doubling measure which is not the restriction of a global doubling measure. Let us fix $s > 0$ and define $d\mu_\Omega = \delta_\Omega^s dx$ for any proper non-empty open subset $\Omega$ of $\mathbb{R}^n$. Note that $\mu_\Omega \in D_t(\Omega)$ for every proper open subset $\Omega$ of $\mathbb{R}^n$, with doubling constant dependent only on $s, n$, and $t$. We define $\Omega_k$ to consist of the interval $(0, 2)$ with the points $i/k$ removed, $1 \leq i \leq k$. Suppose that $\mu_{\Omega_k}$ is a restriction of a global doubling measure $\mu_k$. Since the measure of a countable set is zero (for any global doubling measure), $\mu_k(1, 2)/\mu_k(0, 1) \to \infty$ as $k \to \infty$. By piecing together sets like $\Omega_k$, it is thus easy to define a set $\Omega$ such that $\mu_\Omega$ is not the restriction of a global doubling measure. We could for instance take $\Omega$ to be the bounded open set given by

$$
\Omega = \{2^{-k-1}x + 1 - 2^{-k+1} : x \in \Omega_k, k \in \mathbb{N}\}.
$$

One can even define a domain $D \subset \mathbb{R}^n, n > 1$, such that $\delta_D^s dx$ is strong doubling but not the restriction of a global doubling measure. For instance, if $\Omega$ is as above, then $D = \Omega \times (0, 1) \cup (-1, 0] \times (0, 1)$ is one such domain. Note that here we need the rather well-known fact that line segments are null sets for all doubling measures on $\mathbb{R}^2$; this fact is, for example, an easy corollary of Theorem 2.4).

2. Geometric decay of the measure of a QHBC boundary layer

In this section, we shall prove that the measure of the boundary layer of a QHBC domain decays like a power of its thickness if the measure is appropriately strong doubling. We begin, though, with some preliminary definitions and lemmas. If $p$ is an exponent and $S$ is a set, we write $p' = p/(p-1)$, and $\chi_S$ for the characteristic function of $S$. If $\Omega$ is a bounded domain, we denote by $\text{diam}(\Omega)$ and $\text{inrad}(\Omega)$ its diameter and inradius (the latter being the radius of the largest ball that fits inside $\Omega$). If $t > 0$ and $f \in L_{\text{loc}}^1(\Omega)$, we define the maximal function

$$
M_t f(x) \equiv M_{t, \Omega, f}(x) = \sup_{x \in B_{\subset \Omega}} \frac{1}{\mu(tB)} \int_{tB \cap \Omega} |f| \, d\mu,
$$

where the supremum is taken over all balls $B$ satisfying the indicated conditions.

Our first lemma is both a generalization of the well-known Besicovitch Covering Theorem, and a special case of a theorem of Morse [M] (also stated in [G]), and consequently needs no proof.
Lemma 2.1. Suppose that $0 < s < 1$, that $A \subset \mathbb{R}^n$, and that $\mathcal{F}$ is a family of balls of bounded radius. If for every $x \in A$, $\mathcal{F}$ contains a ball $B_x$ of radius at most $R$ such that $x \in sB_x$, then there exist subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k \subset \mathcal{F}$ such that each $\mathcal{F}_i$ is a pairwise disjoint collection of balls, $\bigcup_{i=1}^k \mathcal{F}_i$ covers $A$, and $k \leq N$ for some $N$ dependent only on $n$ and $s$.

The following lemma belongs to the large family of results that state that various maximal operators are bounded on $L^p$, $1 < p \leq \infty$.

Lemma 2.2. If $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $\mu$ is a positive Borel measure on $\mathbb{R}^n$, then $M_t \mu$ is bounded on $L^p(\Omega, \mu)$ for all $1 < p \leq \infty$, $1 < t$. Furthermore, its operator norm is bounded by $Cp'$, for some constant $C$ dependent only on $n$ and $t$.

Note that we do not assume that $\mu$ satisfies any doubling assumption. If we assumed that $\mu \in D_\delta(\Omega)$, then the alternative “5-covering lemma” (see e.g. [S1, Section 1.1]) could be used in place of Lemma 2.1 in the following proof sketch; additionally, the lemma would be true for all $t > 0$, and not just $t > 1$.

Sketch of proof of Lemma 2.2. As usual for results of this type, the proof consists of an interpolation between the (obvious) boundedness of $M_t$ on $L^\infty(\Omega, \mu)$, and its boundedness from $L^1(\Omega, \mu)$ to the Lorentz (or “weak-type”) space $L^{1,\infty}(\Omega, \mu)$. Such weak-type boundedness results are always proved by means of a covering theorem (see, for example, [S2, Section 1.3.1]). Here, we take $f \in L^1(\Omega, \mu)$, fix a cut-off value $\alpha > 0$ and, for each $x$ such that $A = \{y : M_t f(x) > \alpha\}$, we associate a ball $B'_x$ such that $x \in B'_x \subset \Omega$, and such that the $\mu$-average of $|f|$ on $B_x \equiv tB'_x$ exceeds $\alpha$. By applying Lemma 2.1 with $s = 1/t$ to the family $\{B_x : x \in A\}$, weak boundedness follows in the usual manner. \qed

The next lemma is also a variant of a rather well-known lemma (e.g. see [Bo]); we include a proof for completeness. In its proof and later, we use $A \lesssim B$ if $A \leq CB$ for some constant $C$ dependent only on allowed parameters. In particular, we stress that $C$ is not allowed to depend on $p$ in this lemma.

Lemma 2.3. Suppose that $1 \leq p < \infty$, $1 < t$, $\Omega \subset \mathbb{R}^n$, and $\mu \in D_\delta(\Omega)$. Let $\mathcal{F}$ be a family of balls contained in $\Omega$, and let $a_B$ be a non-negative number for each $B \in \mathcal{F}$. Then

$$\left\| \sum_{B \in \mathcal{F}} a_B \chi_{tB} \right\|_{L^p(\Omega, \mu)} \leq C_p \left\| \sum_{B \in \mathcal{F}} a_B \chi_B \right\|_{L^p(\Omega, \mu)},$$

where $C$ depends only on $n$, $t$, and $C_{\mu, t}$.

Proof. Let $g$ be a non-negative function in $L^{p'}(\Omega, \mu)$. Since $\mu$ is $t$-doubling,

$$A \equiv \int_\Omega \left( \sum_{B \in \mathcal{F}} a_B \chi_{tB} \right) g^t \mu \lesssim \sum_{B \in \mathcal{F}} a_B \left[ \frac{1}{\mu(tB)} \int_{tB \cap \Omega} g \, d\mu \right] \cdot \mu(B).$$

\
We now use the fact that the bracketed quantity is dominated by $M_t g(x)$ for every $x \in B$, together with Hölder’s inequality and Lemma 2.2, to get

$$A \lesssim \sum_{B \in F} a_B \int_B M_t g \, d\mu = \int_{\Omega} M_t g \cdot \sum_{B \in F} a_B \chi_B \, d\mu$$

$$\leq \| M_t g \|_{L^p(\Omega, \mu)} \cdot \left\| \sum_{B \in F} a_B \chi_B \right\|_{L^p(\Omega, \mu)}$$

$$\lesssim \| g \|_{L^p(\Omega, \mu)} \cdot \left\| \sum_{B \in F} a_B \chi_B \right\|_{L^p(\Omega, \mu)}$$

Taking a supremum over all $g \geq 0$ in the unit ball of $L^p(\Omega, \mu)$, the required result follows by duality. \(\square\)

In [SS], Smith and Stegenga prove that if $\Omega \subset \mathbb{R}^n$ is a QHBC domain, then the Minkowski dimension $d$ of $\partial \Omega$ is bounded away from $n$, i.e. the Lebesgue measure of the “boundary layer” decays geometrically; for more on Minkowski content and the decay of the Lebesgue measure of boundary layers of sets, we refer the reader to [MV]. In the planar simply-connected case, Smith and Stegenga’s result follows, with a sharp estimate of $d$, from the results in [JM]. Koskela and Rohde [KR], reproved Smith and Stegenga’s result, in the process getting the sharp estimate of $d$ in all dimensions. The next theorem generalizes this boundary layer decay to the setting of strong doubling measures; our proof is based on the method of [KR].

**Theorem 2.4.** Suppose that $\Omega$ is QHBC and that $\mu \in D_t(\Omega)$, for some $t > t_0$, where $t_0 \in (1, \infty)$ is dependent only on $n$ and $C_{\Omega}$. Then there exist $C, \alpha > 0$ dependent only on $n, C_{\Omega}$, and $C_{\mu,t}$, such that

$$\mu(\Omega_r) \leq C(r/ \text{diam}(\Omega))^\alpha \mu(\Omega) < \infty, \text{ for all } r > 0.$$

In the above statement, recall that $C_{\Omega}$ is the QHBC constant of $\Omega$ and $C_{\mu,t}$ is the $t$-doubling constant of $\mu$. The QHBC condition is necessary in the above theorem—just take $\mu$ to be Lebesgue measure, and $\Omega \subset \mathbb{R}^n$ to be any domain whose boundary has Minkowski dimension $n$. It is also necessary to assume a $D_t(\Omega)$ condition for sufficiently large $t$. For instance, if $\Omega \subset \mathbb{R}^2$ consists of all points in the unit disk whose argument is at most $\theta \in (0, \pi)$, the measure $d\mu(x) = (|x| \log^2(2/x))^{-1} dx$ does not satisfy the conclusion of the theorem even though $\mu \in D_t(\Omega)$ for $t < \sec^{-1}\theta$ (note that both $\sec^{-1}\theta$ and $C_{\Omega}$ tend to infinity as $\theta \to 0$).

**Proof of Theorem 2.4.** Assuming $t \geq t_1 \equiv \text{diam}(\Omega)/\text{inrad}(\Omega)$, the doubling condition ensures that $\mu(\Omega) < \infty$; note also that $t_1$ is bounded above by a constant dependent only on $C_{\Omega}$. Without loss of generality, we normalize $\Omega$ so that $\text{diam}(\Omega) = 1,$
and $\mu$ so that $\mu(\Omega) = 1$. Let $\epsilon = 1/C_\Omega$ and $c = 1/10$. For each $x \in \partial \Omega$, and $n > 0$, we define

$$A_n(x) = \{ y \in \mathbb{R}^n : (1 + \epsilon)^{-n} < |x - y| < (1 + \epsilon)^{-n+1} \},$$

$$\chi_n(x) = \begin{cases} 1, & \text{if } \exists y \in \Omega \cap A_n(x) : d(y, \partial \Omega) > \alpha |x - y| \\ 0, & \text{otherwise}, \end{cases}$$

$$\sigma_n(x) = \sum_{k=1}^{n} \chi_k(x).$$

Koskela and Rohde [KR] prove that the boundary of a QHBC domain is what they term an $\epsilon$-mean porous set (with auxiliary constant $c = 1/10$, as here). This means that there exists a number $n_0$, depending only on $C_\Omega$, such that $\sigma_n(x) > n/2$ for all $n \geq n_0$ (actually, the mean porosity of a set only implies the existence of certain holes in its complement, but an examination of the proof of Theorem 5.1 in [KR] reveals that one can assume that these holes are contained in the domain itself, as we do here).

It follows, as in Theorem 2.1 of [KR], that we can find a collection $\mathcal{F}$ of pairwise disjoint open balls and constants $t_2 > 1$, $j_0 \geq 1$, $c' > 0$, all dependent only on $n$ and $C_\Omega$, such that

$$\sum_{B \in \mathcal{F}} \chi_{t_2B}(x) > c' j, \quad x \in \Omega_{2^{-j}}, \; j \geq j_0.$$

Note that in [KR], an initial reduction argument (which we do not use here) gives $n_0 = 1$, and hence $j_0 = 1$. We define $t_0 = \max \{ t_1, t_2 \}$.

Writing $u(x) = \sum_{B \in \mathcal{F}} \chi_{t_2B}(x)$ for all $x \in \Omega$, we have $\exp(au(x)) > \exp(ac'j)$ for all $x \in \Omega_{2^{-j}}$, $j > j_0$, and $a > 0$. It therefore suffices to find a constant $a = a(n, C_\Omega, C_\mu, t)$ such that

$$\int_{\Omega_{2^{-j}}} e^{au(x)} \, d\mu(x) \lesssim \mu(\Omega_1).$$

Now,

$$\int_{\Omega_{2^{-j}}} e^{au} \, d\mu \leq \sum_{k \geq 0} \int_{\Omega_1} \frac{(au)^k}{k!} \, d\mu \leq \mu(\Omega_1) + \sum_{k > 0} \frac{a^k}{k!} \int_{\Omega_1} \left( \sum_{B \in \mathcal{F}} \chi_{t_2B} \right)^k \, d\mu.$$

Since $\mu \in D_t(\Omega) \subset D_{t_2}(\Omega)$, we may use Lemma 2.3 to get

$$\int_{\Omega_{2^{-j}}} e^{au(x)} \, d\mu(x) \leq \mu(\Omega_1) + \sum_{k > 0} \frac{(aCk)^k}{k!} \int_{\Omega_1} \left( \sum_{B \in \mathcal{F}} \chi_B \right)^k \, d\mu$$

$$\lesssim \mu(\Omega_1) \left( 1 + \sum_{k > 0} \frac{(aCk)^k}{k!} \right).$$

This last series converges for all $a < 1/C\epsilon$, and so we are done. $\square$
REFERENCES


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