Radial variation in some function spaces

David Walsh

(Communicated by Sten Kaijser)

2000 Mathematics Subject Classification. 30H05, 31A05, 46E15.
Keywords and phrases. Radial variation, Besov space, Lipschitz space.

Abstract. In a previous paper [8] we considered properties of the radial variation of analytic functions in a class of Besov spaces $A_{pq}^{s}$, $s > 0$. Here we wish to extend these results to certain related spaces. These are the Lipschitz classes $\Lambda_{s}$ and the mean Lipschitz classes $\Lambda_{p,s}$ where $p \geq 1, 0 < s < 1$. We also consider $A_{pq}^{s}$, where $s = 0$, although the results obtained for these are not as good as when $s > 0$.

1. Introduction

If $f$ is analytic in the disc, the radial variation function of $f$ is the function defined on the disc by

$$F(r,t) = \int_{0}^{r} |f'(ue^{it})| \, du, \quad r < 1, \quad 0 \leq t \leq 2\pi.$$  

Since $f(re^{it}) - f(0) = \int_{0}^{r} f'(ue^{it}) \, du$, it is clear that

$$|f(re^{it})| \leq |f(0)| + F(r,t), \quad r < 1, \quad 0 \leq t \leq 2\pi,$$
and \( F(r, t) \) is a majorant for \( f \). The function \( F(r, t) \) represents the length of the image of the radius vector \([0, re^{it}]\) under the mapping \( f \). It is clear from the definition, that the boundary function \( F(t) = \lim_{r \to 1} F(r, t) \) exists, finite or infinite, for all \( t \in [0, 2\pi] \). It is known as the radial or total variation. An immediate property of \( F \) is that if \( F(t) < \infty \), then \( \lim_{r \to 1} f(re^{it}) \) exists.

We saw in [8] that the property that \( f \in A^s_{pq} \), \( 0 < s < 1, 1 \leq p, q < \infty \), translated into meaningful results for \( F \), in particular that \( F(r, t) \) satisfies an analogous condition on the disc. In Section 1 we are led naturally to consider the case \( s = 0 \) when we ask for a condition under which \( F(t) \) is an integrable function on the circle. It follows immediately that \( F \in L^1 \) if and only if \( f \in A^{1}_{11} \). We then show that \( F(r, t) \) satisfies a corresponding condition to that by \( f \) in the disc. This result extends to the general case \( f \in A^0_{pq} \). In Section 3 we suppose that \( f \) belongs to a Lipschitz space or a mean Lipschitz space and show that both \( F(r, t) \) and \( F(t) \) exhibit the expected behaviour.

1.1 Preliminaries. Let \( D \) denote the unit disc, \( T \) the unit circle in the complex plane and \( L^p = L^p(T) \) the usual Lebesgue space when \( 0 < p < \infty \). For \( p \geq 1 \) we denote the norm of a function \( f \in L^p \) by \( ||f||_p \). For convenience we shall let \( m \) denote normalised Lebesgue measure on the circle \( T \).

Let \( \Delta_t f(z) = f(z+ti) - f(z) \) and \( \Delta_t^m = \Delta_t(\Delta_t^m)^{-1} \). For \( 0 < s \leq 1 \), the Lipschitz class \( \Lambda_s \) is the space of \( 2\pi \)-periodic functions on \([−\pi, \pi]\) for which \( |\Delta_t f(z)| = O(|t|^s) \) uniformly in \( x \). A generalization is the mean Lipschitz class \( \Lambda(p, s) \) consisting of all functions \( f \) for which \( ||\Delta_t f||_p = O(|t|^s) \) for \( t > 0 \); \( \Lambda(p, s) \) reduces to \( \Lambda_s \) when \( p = \infty \). Suppose now that \( f \) is analytic in \( D \). If \( 0 \leq r < 1 \), let

\[
M_p(f, r) = \left( \int_{-\pi}^{\pi} |f(re^{it})|^p \ dm \right)^{1/p}, \quad (0 < p < \infty),
\]

denote the integral mean of \( f \) of order \( p \). It is well known that \( M_p(f, r) \) is an increasing function of \( r \) on \([0, 1] \) and that the class of functions \( f \) for which \( \sup_{r<1} M_p(f, r) < \infty \), is the familiar Hardy space \( H^p \) [2]. For \( 1 \leq p, q < \infty, s > 0 \), and an arbitrary integer \( m > s \), we define the Besov space \( B^s_{pq} \) by

\[
B^s_{pq} = \left\{ f \in L^p : \int_{-\pi}^{\pi} \frac{||\Delta_t^m f(t)||_q}{|t|^{1+sq}} \ dm(t) < \infty \right\}.
\]

It is well known that the definition is independent of \( m \). For a discussion of these spaces see [1], [3], [4], [6], [7]. When \( s \) passes through a positive
integer value, the working definition of the Besov space $B^s_{pq}$ may require a change as indicated above.

The previous definition is no longer valid when $s \leq 0$; for these cases another description is required. For $n \geq 1$ we let $W_n$ be the polynomial on $T$ whose Fourier coefficients satisfy $\hat{W}_n(2^n) = 1$, $\hat{W}_n(j) = 0$ for $j \notin (2^{n-1}, 2^n+1)$ and $W_n$ is a linear function on $[2^{n-1}, 2^n]$ and on $[2^n, 2^{n+1}]$. If $n < 0$ we put $W_n = W_{-n}$. We put $W_0 = z + 1 + z$. For $s \leq 0$, $1 \leq p, q < \infty$, $B^s_{pq}$ consists of all distributions $f$ on $T$ for which

$$\sum_{n=-\infty}^{\infty} 2^n |s| \|f \ast W_n\|_p^q < \infty.$$ 

It is known that this description is equivalent to the previous one for $s > 0$, but for $s = 0$ in particular, only the second definition is valid. See [4] Appendix 2, [1]. In fact when $q > p$ there exist $f \in B^0_{pq}$ such that $f \notin L^p$.

Let $A^s_{pq}$ denote the subspace of $B^s_{pq}$ consisting of analytic functions. The space $A^s_{pq}$ for $s > 0$, may be characterized as follows: for an arbitrary integer $m > s$ the analytic function $f \in A^s_{pq}$ if and only if

$$\|f\|_A = |f(0)| + \left\{ \int_0^1 (1 - r^2)^{q(m-s)-1} M_p(f^{(m)}, r)^q r dr \right\}^{1/q} < \infty.$$ 

Once again the definition is independent of $m$ for $m > s$. For $s = 0$ this definition is easily modified. This is because of the property that $f \in A^0_{pq}$ if and only if $If \in A^1_{pq}$ where $I$ is the integration operator. Therefore $f \in A^0_{pq}$ if and only if with $m = 2$,

$$\|f\|_A = |f(0)| + \left\{ \int_0^1 (1 - r^2)^{q-1} M_p(f', r)^q r dr \right\}^{1/q} < \infty,$$

and with $m = 3$, if and only if

$$\|f\|_A = |f(0)| + \left\{ \int_0^1 (1 - r^2)^{2q-1} M_p(f^{(2)}, r)^q r dr \right\}^{1/q} < \infty.$$ 

We shall need both of these representations. In particular with $p = q = 1$ we have $f \in A^0_{11}$ if and only if

$$\|f\|_A = |f(0)| + \int_0^1 \int_0^{2\pi} |f'(re^{it})| \, dr \, d\theta < \infty.$$
2. Integrability of $F$

The function $F(t) = F(1, t)$ is given from (1) by

$$F(t) = \int_0^1 |f'(ue^{it})| \, du, \quad 0 \leq t \leq 2\pi.$$  

We now ask what is a sufficient condition that $F \in L^1$? Since $F \in L^1$ if and only if $\int_0^{2\pi} \int_0^1 |f'(re^{it})| r \, dr \, dm < \infty$, the answer is immediate from the definition:

**Proposition 1.** $F \in L^1(T)$ if and only if $f \in A_{11}^0$. Moreover

$$\|F\|_1 + |f(0)| = \|f\|_A.$$  

It may be observed here that if $f \in A_{11}^0$ then its boundary function $f(e^{it})$ exists a.e.; in fact $f \in H^1$. This follows by integrating the obvious inequality $|f(re^{it})| \leq |f(0)| + \int_0^r |f'(ue^{it})| \, du$.

We can equally express the relationship in terms of the $A$-norm of $F(r, t)$. For this purpose we introduce the gradient of $F$: $\nabla F(r, t) = (\partial F/\partial r, 1/r \partial F/\partial t) = (|f'(re^{it})|, 1/r \partial F/\partial t)$. The relationship referred to is

$$f \in A_{11}^0 \quad \text{if and only if} \quad \int_0^1 \int_0^{2\pi} |
abla F| \, dr \, dm < \infty.$$  

If the integral is finite then it follows very simply that $f \in A_{11}^0$ and that $\|f\|_A \leq |f(0)| + \|F\|_A$. The proof in the other direction has already been done in essence in [8] where we considered only $s > 0$. In fact we can state a more general result which follows from Theorem 1 there, and which works without any changes for our situation.

**Theorem 1.** Suppose that $1 \leq p, q < \infty$. There is a constant $C = C(p, q)$ such that if $f \in A_{pq}^0$ then

$$\frac{1}{p} \left( \int_{-\pi}^\pi |\nabla F(r, t)|^p \, dm \right)^{q/p} \, r \, dr \leq C \|f\|_A^q.$$  

**Proof.** The proof in [8] goes through word for word with $s = 0$. In the case $p = q = 1$ it is simpler since the use of Hölder’s inequality is not needed. We do make use of the alternative definitions of $A_{pq}^0$ mentioned above.  

$\square$
If the double integral for $F(r,t)$ is finite then as noted already it is clear that $f \in A_{pq}^0$. The question when $F \in L^p$, $p > 1$, does not have so neat an answer. A reasonable sufficient condition is given by

**Theorem 2.** Suppose that $1 \leq p, q < \infty$. If $f \in A_{pq}^0$ then

$$
\|F\|_p \leq \|f\|_A.
$$

**Proof.** By Minkowski’s Inequality in continuous form

$$
\left( \int_0^{2\pi} |F(t)|^p \, dm \right)^{1/p} = \left( \int_0^{2\pi} \left( \int_0^1 \left| f'(re^{it}) \right| \, dr \right)^p \, dm \right)^{1/p}
$$

$$
\leq \int_0^1 \left( \int_0^{2\pi} \left| f'(re^{it}) \right|^p \, dm \right)^{1/p} \, dr,
$$

$$
< \infty,
$$

and $\|F\|_p \leq \|f\|_A$. \hfill \Box$

**Remark.** The condition $f \in A_{pq}^0$ implies that $f \in H^p$ for all $p \geq 1$. To see this we note that for $r < 1$

$$
|f(re^{it})| \leq |f(0)| + \int_0^r |f'(ue^{it})| \, du.
$$

On using Minkowski’s Inequality again we obtain

$$
M_p(f,r) \leq |f(0)| + \int_0^r M_p(f',u) \, du
$$

$$
\leq \|f\|_A
$$

and the result is immediate.

In [8] it was shown that if $f \in A_{pq}^s$, $0 < s < 1$, then the boundary function $F \in B_{pq}^s$. We do not know whether this is true for the case $s = 0$ since the proof given there is no longer valid.

### 3. The Lipschitz spaces

The Lipschitz space $\Lambda_s$, $0 < s < 1$, may be regarded as the Besov space $B_{\infty\infty}^s$. It is well known that for an analytic function $f$ on the disc, $f \in \Lambda_s$ if and only if there exists $M$ such that

$$
|f'(z)| \leq \frac{M}{(1-r)^{1-s}}
$$
This property has its counterpart for the function $F(r, t)$.

**Theorem 3.** The function $f \in \Lambda_s$, $0 < s < 1$, if and only if $\nabla F(r, t) = O((1 - r)^{s-1})$.

**Proof.** Suppose $f \in \Lambda_s$ and let $M$ be the number noted above. First we show that $F(t)$ is bounded.

$$F(r, t) = \int_0^r |f'(ue^{it})| \, du \leq M \int_0^r \frac{1}{(1-u)^{1-s}} \, du = M (1 - (1 - r)^s) / s \leq M / s,$$

for all $r < 1$ and so $F(t)$ is bounded.

Since the first component of $\nabla F(r, t)$ is $|f'(re^{it})|$ we need only consider the second. Now by Lemma 3 of [8], $\frac{\partial F}{\partial t}(r, t) = \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it}) \, du$ and

$$\left| \frac{1}{r} \frac{\partial F}{\partial t}(r, t) \right| = \left| \frac{1}{r} \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it}) \, du \right|
\leq \frac{1}{r} \int_0^r u |f''(ue^{it})| \, du
\leq M \int_0^r \frac{1}{(1-u)^{2-s}} \, du \leq M' \frac{1}{(1 - r)^{1-s}}.$$

In the second inequality above we used Theorem 5.5 of [2]. The result follows. \qed

There is a corresponding result for $F(t)$.

**Theorem 4.** If $f \in \Lambda_s$, $0 < s < 1$, then $F(t) \in \Lambda_s$.

**Proof.** We have shown that $F$ is bounded. We write

$$F(x) - F(t) = F(x) - F(r, x) + F(r, x) - F(r, t) + F(r, t) - F(t).$$

But

$$F(x) - F(r, x) = \int_r^1 |f'(ue^{ix})| \, du \leq M \int_r^1 \frac{1}{(1-r)^{1-s}} \, du \leq M (1 - r)^s / s$$

and the same holds for $F(r, t) - F(t)$. Moreover $F(r, x) - F(r, t) = \int_t^x \frac{\partial F}{\partial v}(r, v) \, dv$. Consequently
\[ |F(r, x) - F(r, t)| \leq \left| \int_t^x \frac{\partial F}{\partial v}(r, v) \, dv \right| \leq M' \left| \int_t^x \frac{1}{(1-r)^{1-s}} \, dv \right| = M' \frac{1}{(1-r)^{1-s}} |t - x|, \]

on using the previous theorem. If we now choose \( 1 - r = |x - t| \) we get

\[ |F(r, x) - F(r, t)| \leq M'' |t - x|^s \]

and \( F(t) \in \Lambda_s \). \( \square \)

The mean Lipschitz classes \( \Lambda_{p,s}(T) \), \( 1 \leq p \), \( 0 < s < 1 \), are identical with the Besov spaces \( B_{p,\infty}^s \). They satisfy the condition: A function \( g \in L^p(T) \) belongs to \( \Lambda_{p,s} \) if

\[ \|g\|_{p,s} = \left( \int_0^{2\pi} |g(x + t) - g(x)|^p \, dx \right)^{1/p} = O(|t|^s) \]

for small \( t \). It is known (Theorem 5.4 of [2]) that an analytic function \( f \) is in \( \Lambda_{p,s} \) if and only if \( M_p(f', r) = O\left( \frac{1}{(1-r)^s} \right) \quad 0 < r < 1 \). With the aid of this, similar results to those of the last two theorems can be shown to hold and the proofs are straightforward.

**Theorem 5.** If \( f \in \Lambda_{p,s} \), \( 1 \leq p \), \( 0 < s < 1 \), then there exists \( C = C(p, s) \) such that

(a) \( \left( \int_{-\pi}^\pi |\nabla F(r, t)|^p \, dm \right)^{1/p} \leq C \|f\|_{p,s} (1-r)^{s-1}; \)

(b) \( F(t) \in \Lambda_{p,s} \) and \( \|F\|_{p,s} \leq C \|f\|_{p,s} \).

Whether a particular type of continuity for \( f \) implies the same holds for \( F \) is uncertain. The boundary function \( f(e^{it}) \) is absolutely continuous if and only if \( f' \in H^1 \). We don’t know that this implies that \( F(t) \) is absolutely continuous but it does imply that \( F \) is continuous.

**Proposition 2.** If \( f(e^{it}) \) is absolutely continuous then \( F(t) \) is continuous.

**Proof.** We have \( F(t + x) - F(t) = \int_0^1 (|f'(re^{i(t+x)})| - |f'(re^{it})|) \, dr \) and therefore

\[ |F(t + x) - F(t)| \leq \int_0^1 |f'(re^{i(t+x)}) - f'(re^{it})| \, dr \]

\[ = \int_0^1 |f'(re^{it}) - f'(re^{it})| \, dr \]
where $g_x(t) = g(t+x)$ is a translate of $g$. The Fejer-Riesz inequality allows us to conclude

$$|F(t+x) - F(t)| + |F(t + x + \pi) - F(t + \pi)| 
\leq \int_{-1}^{1} |f_x'(re^{it}) - f'(re^{it})| \, dr \leq \frac{1}{2} \int_{0}^{2\pi} |f_x'(re^{it}) - f'(re^{it})| \, dx \to 0$$

as $x \to 0$ uniformly in $t$, because the translation map $x \to g_x$ is uniformly continuous from $T$ to $L^1$. The proof is complete. □

In [8] it was seen that if we assume slightly more, namely if $f \in A^{11}_1$, then $F \in B^{11}_1$ which implies that $F$ is absolutely continuous. However mere continuity of $f$ on the circle does not even imply that $F$ is bounded. In fact Walter Rudin [5] has shown that there exists an analytic function $f$ continuous in the closed disc, such that $F(t) = \infty$ almost everywhere.

References