On the Number of Polynomials with Small Discriminants in the Euclidean and $p$-adic Metrics

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Abstract  In this article it is proved that there exist a large number of polynomials which have small discriminant in terms of the Euclidean and $p$-adic metrics simultaneously. The measure of the set of points which satisfy certain polynomial and derivative conditions is also determined.

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1 Introduction and Main Results

In this paper the distribution of the discriminants of integer polynomials is investigated. In particular, a lower bound for the number of polynomials which have small discriminant in both the Euclidean and $p$-adic metrics is determined. Since, the $p$-adic norm of these discriminants is small they are clearly divisible by large powers of $p$. This gives some information regarding the distribution of the roots of polynomials and shows that a large number of integer polynomials have roots which are simultaneously close in the $p$-adic and Euclidean norms. These and related questions were first introduced and studied by Mahler [1] in 1964. Other results (detailed below) have been separately proved for the real [2] and $p$-adic [3] fields. More information regarding root separation for integer polynomials may be found in [4–7] and [8].

First some notation is needed. Throughout this paper,

$$P(f) = a_nf^n + \cdots + a_1f + a_0$$

is an integer polynomial with degree $\deg P = n$ and height $H = H(P) = \max_{0 \leq j \leq n} |a_j|$. Let $\mu_1(A_1)$ be the Lebesgue measure of a measurable set $A_1 \subset \mathbb{R}$, and $\mu_2(A_2)$ the Haar measure of a measurable set $A_2 \subset \mathbb{Q}_p$. Using these definitions, define the product measure $\mu$ on $\mathbb{R} \times \mathbb{Q}_p$ by setting $\mu(A) = \mu_1(A_1)\mu_2(A_2)$ for a set $A = A_1 \times A_2$. The cardinality of a set $S$ will be denoted by $\#S$. We will use the Vinogradov symbols $\ll$ (and $\gg$) where $a \ll b$ implies that there exists a constant $C > 0$ such that $a \leq Cb$. If $a \ll b \ll a$ then we write $a \asymp b$.

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Let $\alpha_1, \ldots, \alpha_n$ be the complex roots of the polynomial $P \in \mathbb{Z}[x]$. The discriminant of $P$, denoted by $D(P)$ is defined as

$$D(P) = \alpha_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.$$ 

Alternatively, $D(P)$ can be defined as the determinant of a matrix containing only the coefficients of $P$. Hence $D(P) \in \mathbb{Z}$ and if $P$ does not have multiple roots then

$$1 \leq |D(P)| \ll H(P)^{2n-2}.$$

Consider the set of polynomials

$$P_n(Q) = \{ P \in \mathbb{Z}[x] : \deg P \leq n, H(P) \leq Q \}$$

and note that the cardinality of this set is comparable to $Q^{n+1}$. Finally, let $v_1, v_2 \in \mathbb{R}^+ \cup \{0\}$ and define the set of polynomials

$$P_n(Q, v_1, v_2) = \{ P \in P_n(Q), 1 \leq |D(P)| < Q^{2n-2v_1}, |D(P)|_p < Q^{-2v_2} \},$$

where $\cdot |_p$ is the standard $p$-adic valuation. For this article we will consider $P_n(Q, v_1, v_1)$ and for simplicity we will write $P_n(Q, v_1) = P_n(Q, v_1, v_1)$.

**Theorem 1.1** Let $n \geq 3$ and $0 \leq v_1 < 1/3$ and let $Q_0(n) \in \mathbb{R}$ be a large constant. Then

$$\#P_n(Q, v_1) \gg Q^{n+1-4v_1} \text{ for all } Q > Q_0.$$ 

In [2] it was proved that $\#P_n(Q, v_1, 0) \gg Q^{n+1-2v_1}$ and in [3] that $\#P_n(Q, 0, v_2) \gg Q^{n+1-2v_2}$. These results come from metric theorems of Diophantine approximation in the real and $p$-adic fields respectively. To prove Theorem 1.1 it is necessary to prove a metric theorem in simultaneous Diophantine approximation in $\mathbb{R} \times \mathbb{Q}_p$. For $n = 2$ the discriminant has the form $D(P) = a_1^2 - 4a_0a_2$ and the estimates can be calculated directly as follows. Define $v$ such that $p^{-v} < Q^{-2v_1} \leq p^{-v_1+1}$. Choose $a_2$, with $0 < a_2 < Q$ such that $p \nmid a_2$ and fix $a_1$. Then, there exists $0 < s < p^v$ such that for $a_0 \equiv s \pmod{p^v}$ the linear congruence $4a_0a_2 \equiv a_1^2 \pmod{p^v}$ is satisfied. For any such triple $(a_0, a_1, a_2)$ we have $|D(P)|_p \leq p^{-v} < Q^{-2v_1}$. It remains to count the integers $t$ such that $a_0 = s + tp^v$ and $|a_1^2 - 4a_2a_0| < Q^{-2v_1}$. From this, $t$ must lie in an interval of length at least $Q^{2-2v_1}/(4a_2p^v)$ which implies that there are at least $Q^{1-4v_1}$ such $t$ and therefore such $a_0$. Thus $\#P_2(Q, v_1) \gg Q^{3-4v_1}$.

From now on we assume that $n \geq 3$. Fix a set $I \times K$ where $I$ is an interval contained in $[0, 1) \subset \mathbb{R}$ and $K$ is a cylinder contained in $\mathbb{Z}_p$. Define the set $L_n = L_n(v_0, v_1, v_0, Q)$ to be the set of $(x, w) \in I \times K$ such that the inequalities

$$|P(x)| < c_0Q^{-v_0}, \quad |P(w)|_p < c_0Q^{-v_0}$$

and

$$\delta_0Q^{1-v_1} < |P'(x)| < c_0Q^{1-v_1}, \quad \delta_0Q^{-v_1} < |P'(w)|_p < c_0Q^{-v_1}$$

hold for some $P \in P_n(Q)$. Theorem 1.1 will follow from Theorem 1.2 below.

**Theorem 1.2** Let $n \geq 3$, $v_0 + v_1 = n/2$ and $0 \leq v_1 < 1/3$. For all real numbers $\kappa$ such that $0 < \kappa < 1$ there exist constants $\delta_0$ and $c_0$ such that

$$\mu(L_n(v_0, v_1, c_0, \delta_0, Q)) > \kappa \mu(I \times K)$$

for $Q$ sufficiently large.
It can be readily verified using Dirichlet’s box principle that if \( c_0 = (n + 1)^{3/4} \) then the upper bounds in (1.1) and (1.2) hold for all \( (x, w) \in I \times K \). The main difficulty of this paper is to prove the existence of \( \delta_0 \).

2 Preliminary Results

The following two lemmas show that there is no loss of generality in proving the theorems for the set of irreducible, primitive polynomials \( P \) which satisfy

\[
H(P) \ll |a_n|, \quad |a_n|_p \gg 1.
\]  

(2.1)

Let \( \mathcal{P}_n(Q) \) denote the set of such polynomials with height \( H \leq Q \) and degree at most \( n \). The first lemma was proved in [9].

**Lemma 2.1** Let \( E(x, w) \) be the set of \( (x, w) \in \mathbb{R} \times \mathbb{Q}_p \) such that the inequality

\[
|P(x)||P'(w)|_p < H(P)^{-w}
\]

has infinitely many solutions in reducible polynomials \( P \in \mathbb{Z}[x] \) with \( \deg P \leq n \). Then \( \mu(E(x, w)) = 0 \) for \( w > n - 1 \).

The last lemma was proved in [10].

**Lemma 2.2** Let \( p \) be a prime number and \( P \in \mathbb{Z}[x] \) be primitive and irreducible. Let \( C = C(n, p) > 0 \) be a constant. There exists a natural number \( m, 0 \leq m \leq c(n) \), where \( c(n) > 0 \) is a constant depending only on \( n \), with the following property. Let \( F(x) = P(x + m) \) and \( T(x) = x^n F(1/x) \). Then \( T(x) = b_n x^n + \ldots + b_1 x + b_0 \in \mathbb{Z}[x] \) satisfies

\[
|b_n| \gg H(T), \quad |b_n|_p \gg 1.
\]

The transformations to \( F \) and \( T \) preserve the discriminant; i.e., \( D(P) = D(F) = D(T) \) (see [2] for details).

Let \( P \in \mathcal{P}_n(Q) \) have complex roots \( \alpha_1, \ldots, \alpha_n \) and roots \( \gamma_1, \ldots, \gamma_n \) in \( \overline{\mathbb{Q}}_p \), where \( \overline{\mathbb{Q}}_p \) is the smallest field containing \( \mathbb{Q}_p \) and all algebraic numbers. From (2.1), it can be readily verified that

\[
|\alpha_i| \ll 1 \quad \text{and} \quad |\gamma_i|_p \ll 1
\]

for \( i = 1, \ldots, n \); i.e., the roots are bounded (see [11]). Define the sets

\[
S_1(\alpha_j) = \left\{ x \in \mathbb{R} : |x - \alpha_j| = \min_{1 \leq i \leq n} |x - \alpha_i| \right\}, \quad 1 \leq j \leq n,
\]

\[
S_2(\gamma_k) = \left\{ w \in \mathbb{Q}_p : |w - \gamma_k|_p = \min_{1 \leq i \leq n} |w - \gamma_i|_p \right\}, \quad 1 \leq k \leq n.
\]

We will consider the sets \( S_1(\alpha_j) \) and \( S_2(\gamma_k) \) for fixed \( j \) and \( k \). Without loss of generality, we will assume that \( j = k = 1 \). The other roots of \( P \) are reordered so that

\[
|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3| \leq \ldots \leq |\alpha_1 - \alpha_n|,
\]

\[
|\gamma_1 - \gamma_2|_p \leq |\gamma_1 - \gamma_3|_p \leq \ldots \leq |\gamma_1 - \gamma_n|_p.
\]

The next lemma is proved in [11].

**Lemma 2.3** Let \( x \in S_1(\alpha_1) \) and \( w \in S_2(\gamma_1) \) where \( \alpha_1 \) and \( \gamma_1 \) are complex and \( p \)-adic roots of a polynomial \( P \in \mathbb{Z}[x] \), respectively. Then,

\[
|x - \alpha_1| < n|P(x)||P'(x)|^{-1},
\]
\[ |w - \gamma_1|_p < |P'(w)|_p |P'(w)|^{-1}, \]
\[ |x - \alpha_1| < 2^n \min \left( |P(x)||P'(\alpha_1)|^{-1}, |P(x)||P'(\alpha_1)|^{-1}|\alpha_1 - \alpha_2|^{1/2} \right), \]
\[ |w - \gamma_1|_p < \min \left( |P(w)|_p |P'(\gamma_1)|^{-1}, |P(w)|_p |P'(\gamma_1)|^{-1}|\gamma_1 - \gamma_2|^{1/2} \right) \]

hold.

The following theorem [12] will deal with the case of small derivatives.

**Theorem 2.4** ([12, Theorem 1.3]) For any \((x, w) \in I \times K\), there exist a neighbourhood \(W = U \times V \subseteq I \times K\) of \((x, w)\) and a constant \(\lambda > 0\) with the following property: for any \(\delta > 0\) and ball \(B \subseteq W\), there exists a constant \(E > 0\) such that the set

\[ \bigcup_{P \in \mathcal{P}_n(Q)} \{(x, w) \in B : |P(x)| < \delta, |P(w)|_p < \delta, |P'(x)| < K_{\infty}, |P'(w)|_p < K_{p} \} \]

has measure at most \(E\xi^3 \mu(B)\), where \(\xi = \max\{\delta, (\delta^2 Q^{-1} K_{\infty} K_{p})^{\frac{1}{n+1}} \}\).

Using the notation of [12], \(f(t) = (t, t^2, \ldots, t^n)\), \(T_1 = \cdots = T_n = q\), \(R = \mathbb{Z}, g(\mathbb{Z}) = 1\) and \(S = \{p, \infty\}\) so that \#S = 2.

**3 Proof of Theorem 1.1**

Following (2.1) we need only to prove the theorems for \(P \in \mathcal{P}_n(Q)\). Let \((x, w) \in \mathcal{L}_n\). Then, there exists \(P \in \mathcal{P}_n(Q)\) such that (1.1) and (1.2) hold. Let \(x \in S_1(\alpha_1)\) and \(w \in S_2(\gamma_1)\), then from Lemma 2.3, we obtain

\[ |x - \alpha_1| < nc_0^{-1}Q^{v_0 - v_0 - 1} \quad \text{and} \quad |w - \gamma_1| < c_0^{-1}Q^{v_1 - v_0}. \]  

(3.1)

Let \(v_1 < 1/3\) so that from \(v_0 + v_1 = n/2\) we have \(v_0 = 2v_1 + \beta\) which implies that

\[ v_0 - v_1 = v_1 + \beta \]

(3.2)

for some \(\beta > 0\). Develop the polynomial \(P'\) as a Taylor series in the neighborhood of the roots \(\alpha_1\) and \(\gamma_1\). This will be demonstrated for the \(p\)-adic coordinate. Estimating each term of the Taylor series \(P'(w) = \sum_{i=1}^{n}(i!)^{-1}P^{(i)}(\gamma_1)(w - \gamma_1)^{i-1}\) gives

\[ |P^{(j)}(\gamma_1)|_p |w - \gamma_1|^{j-1} \ll Q^{j-1}(v_1 - v_0) < \frac{\delta_0 Q^{-v_1}}{4(n - 2)} \]

for \(j = 3, \ldots, n\) and \(Q\) sufficiently large. The fact that \(P \in \mathbb{Z}[x]\) and (2.2) have been used to obtain the trivial bound \(|P^{(j)}(\gamma_1)|_p \ll 1\). Thus,

\[ \frac{\delta_0 Q^{-v_1}}{2} < |P'(w)|_p < |P'(\gamma_1)|_p < 2|P'(w)|_p < 2c_0Q^{-v_1}. \]

(3.3)

Similarly in the real case, using (3.1) and (3.2), for \(Q\) sufficiently large, we obtain

\[ \frac{\delta_0 Q^{1-v_1}}{2} < |P'(x)|_p < |P'(\alpha_1)| < 2|P'(x)|_p < 2c_0 Q^{1-v_1}. \]

(3.4)

(Again the trivial bound \(|P^{(j)}(\alpha_1)| < Q\) is used for \(j \geq 2\).) Using the facts that \(P'(\alpha_1) = a_n \prod_{i=2}(\alpha_1 - \alpha_i)\) and \(P'(\gamma_1) = a_n \prod_{i=1}(\gamma_1 - \gamma_i)\), the formulae for the discriminants can be
rewritten to obtain
\[
|D(P)| = \left| a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2 \right| = \left| |P'(\alpha_1)|^2 a_n^{2n-4} \prod_{2 \leq i < j \leq n} (\alpha_i - \alpha_j)^2 \right|,
\]
\[
|D(P)|_p = \left| a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\gamma_i - \gamma_j)^2 \right|_p = \left| |P'(\gamma_1)|_p^2 a_n^{2n-4} \prod_{2 \leq i < j \leq n} (\gamma_i - \gamma_j)^2 \right|_p. \tag{3.5}
\]

As all the roots are bounded, it follows from Lemma 2.2, (3.3), (3.4) and (3.5) that
\[
|D(P)| < |P'(\alpha_1)|^2 Q^{2n-4} < Q^{2n-2-2v_1}, \\
|D(P)|_p < |a_n^{2n-4}|_p |P'(\gamma_1)|^2 < Q^{-2v_1}. \tag{3.6}
\]

Thus, for every point \((x, w) \in \mathcal{L}_n\) there exists a polynomial \(P \in \mathcal{P}_n(Q)\) which satisfies (3.6).

Also, for any such point there exists a polynomial \(P\) with roots \((\alpha_i, \gamma_j)\) satisfying the system of inequalities
\[
|x - \alpha_i| < nc_0\delta_0^{-1} Q^{v_1-v_0-1}, \quad |w - \gamma_j|_p < c_0\delta_0^{-1} Q^{v_1-v_0} \tag{3.7}
\]
for \(1 \leq i, j \leq n\). For each pair of roots \((\alpha_i, \gamma_j)\) of \(P\) denote the set of solutions of (3.7) by \(\mathcal{M}_{ij}(P)\). Let \(\mathcal{M}(P) = \bigcup_{\leq i, j \leq n} \mathcal{M}_{ij}(P)\). Let \(s\) be the number of polynomials \(P \in \mathcal{P}_n(Q)\) which satisfy (3.6). By (3.6) and the inequalities
\[
\kappa \mu(I \times K) < \mu(\mathcal{L}_n) < s \mu(\mathcal{M}(P)) < 3^3 n^3 c_0^2\delta_0^{-2} Q^{-2v_0+2v_1-1} < Q^{-2v_0+2v_1-1},
\]
we obtain \(s \gg Q^{2v_0-2v_1+1} = Q^{n+1-4v_1}\). Note that by Theorem 1.2 we may choose \(\kappa\) to be close to 1.

4 Proof of Theorem 1.2

Again, from the arguments in Section 2 we need only to prove the theorem for polynomials which satisfy (2.1). Suppose that for \(\delta_0 > 0\) one or both of the lower bounds in (1.2) does not hold. This defines two sets:
\[\mathcal{L}'_n = \{(x, w) \text{ satisfying (1.1): } |P'(x)| < c_0 Q^{1-v_1}, |P'(w)|_p < \delta_0 Q^{-v_1}\},\]
\[\mathcal{L}''_n = \{(x, w) \text{ satisfying (1.1): } |P'(x)| < \delta_0 Q^{1-v_1}, |P'(w)|_p < c_0 Q^{-v_1}\}.
\]

Then, \(\mathcal{L}_n = (I \times K) \setminus (\mathcal{L}'_n \cup \mathcal{L}''_n)\). It will be demonstrated that \(\mu(\mathcal{L}_n) < \frac{1}{4} \mu(I \times K)\). Similar results can be obtained in exactly the same way for \(\mathcal{L}''_n\). This will obviously imply that \(\mu(\mathcal{L}_n) > \kappa \mu(I \times K)\).

First we deal the case of small first derivatives. Note that since \(v_1 < 1/3\) there exists \(\varepsilon > 0\) such that \(v_1 = 1/3 - \varepsilon\). Choose a real number \(\gamma > 0\) such that \(\gamma < 3\varepsilon/2\) and let \(\mathcal{B}_n\) denote the set of \((x, w) \in \mathcal{L}_n\) satisfying
\[
Q^{1-v_1-\gamma} < |P'(x)| < c_0 Q^{1-v_1}, \quad Q^{-v_1-\gamma} < |P'(w)|_p < \delta_0 Q^{-v_1}. \tag{4.1}
\]

Let \(\mathcal{B}'_n\) be defined by \(\mathcal{B}'_n = \mathcal{B}_n \cup \mathcal{B}_n^{\prime}\). From Theorem 2.4, the measure of \(\mathcal{B}'_n\) tends to zero as \(Q \to \infty\). Hence, for \(Q\) sufficiently large, \(\mu(\mathcal{B}'_n) < \frac{1}{4} \mu(I \times K)\). It remains to be shown that \(\mu(\mathcal{B}_n) < \frac{1}{4} \mu(I \times K)\) for sufficiently small \(\delta_0\).

Assume without loss of generality, that the closest roots of \(P\) to \(x\) and \(w\) are \(\alpha_1\) and \(\gamma_1\), respectively. Estimates for \(|P'(\alpha_1)|\) and \(|P'(\gamma_1)|\) are now obtained. From Lemma 2.3, (1.1)
and (4.1), it follows that
\[ |x - \alpha_1| < nc_0 Q^{v_1 - v_0 - 1 + \gamma}, \quad |w - \gamma_1| < c_0 Q^{v_1 - v_0 + \gamma}. \]

Using Taylor's theorem for \( P'(f) \) and (3.2) the inequalities
\[ \frac{1}{2} |P'(x)| < |P'(\alpha_1)| < 2 |P'(x)|, \quad \frac{1}{2} |P'(w)| < |P'(\gamma_1)| < 2 |P'(w)| \]
can be obtained in the same way as (3.3) and (3.4). Thus, from (4.1)
\[ \frac{1}{2} Q^{1 - v_1 - \gamma} < |P'(\alpha_1)| < 2c_0 Q^{1 - v_1}, \quad \frac{1}{2} Q^{-v_1 - \gamma} < |P'(\gamma_1)| < 2c_0 Q^{-v_1}. \]  \( (4.2) \)

Let \( \sigma(P) \) denote the set of points for which (1.1) and (4.2) hold. Using Lemma 2.3 this set is defined by the inequalities
\[ |x - \alpha_1| < nc_0 Q^{-v_1}|P'(\alpha_1)|^{-1}, \quad |w - \gamma_1| < c_0 Q^{-v_1}|P'(\gamma_1)|^{-1}. \]

Note that \( B_n \subset \bigcup_{P \in \mathcal{P}_n(Q)} \sigma(P) \). We will show that the measure of this union is small.

Choose two real numbers \( u_1 \) and \( u_2 \) with the following properties:
\[ u_1 + u_2 = 1 - 2v_1, \quad v_0 > u_1 > 2v_1 + 2\gamma - 1 \geq v_1 - 1, \quad v_0 > u_2 > 2v_1 + 2\gamma > v_1. \]

(4.3)

That this is possible can be readily verified using the conditions on \( v_1, v_0 \) and \( \gamma \). Then, define the set \( \sigma_1(P) \) as the set of \( (x, w) \) for which the inequalities
\[ |x - \alpha_1| < c_1 Q^{-u_1}|P'(\alpha_1)|^{-1}, \quad |w - \gamma_1| < Q^{-u_2}|P'(\gamma_1)|^{-1} \]
hold for \( c_1 \) to be chosen later. From (4.3) and \( Q \) sufficiently large we have that \( \sigma(P) \subset \sigma_1(P) \).

The polynomial \( P \) is now developed as a Taylor series in \( \sigma_1(P) \) and each term is estimated from above. Only the real coordinate will be demonstrated. We have
\[ |P'(\alpha_1)||x - \alpha_1| < c_1 Q^{-u_1}, \quad \frac{1}{j!} |P^{(j)}(\alpha_1)||x - \alpha_1|^j \ll Q^{1-j(u_1+1-v_1-\gamma)} \]
for \( j = 2, \ldots, n \). The fact that \( |P^{(j)}(\alpha_1)| \ll Q \) was used. Thus, from (4.3), \( |P(x)| \leq 2c_1 Q^{-u_1} \) for \( Q \) sufficiently large. It is similarly possible to estimate \( P'(x) \) on \( \sigma_1(P) \) so that \( |P'(x)| \leq 3c_0 Q^{2-v_1} \). In exactly the same way the inequalities
\[ |P'(w)| \ll 2Q^{-u_2}, \quad |P'(w)| \ll 3\delta_0 Q^{-v_1} \]
can also be obtained.

Let \( b \) be the vector \( (a_1, \ldots, a_2) \) and let \( \mathcal{P}_n^b(Q) \) be the set of polynomials in \( \mathcal{P}_n(Q) \) which have the same vector \( b \). Note that the number of vectors \( b \) is at most \( (2Q + 1)^n \ll (3Q)^n \).

We now use Sprindzuk's method of essential and inessential domains (see [11] for details). A polynomial \( P \in \mathcal{P}_n^b(Q) \) is called essential if \( \mu(\sigma_1(P) \cap \sigma_1(P')) \leq \frac{1}{2} \mu(\sigma_1(P)) \) for all polynomials \( P' \in \mathcal{P}_n^b(Q) \). It is called inessential otherwise. Let \( E_n^b(Q) \) be the set of essential \( P \) and \( I_n^b(Q) \) be the set of inessential \( P \). Thus \( \mathcal{P}_n^b(Q) = I_n^b(Q) \cup E_n^b(Q) \) and
\[ \bigcup_{P \in \mathcal{P}_n^b(Q)} \sigma(P) = \left( \bigcup_{P \in E_n^b(Q)} \sigma(P) \right) \cup \left( \bigcup_{P \in I_n^b(Q)} \sigma(P) \right). \]
First we consider the essential polynomials. Note that \( \mu(\sigma(P)) \leq \frac{2}{c_1} Q^{-2v_0 + v_1 + u_2} \mu(\sigma_1(P)) \). Clearly \( \sum_{P \in \mathcal{P}_{n_0}^{(i)}(Q)} \mu(\sigma_1(P)) \leq 2\mu(I \times K) \). Thus, from (4.3) and the fact that \( v_0 + v_1 = n/2 \), the set of points lying in sets \( \sigma(P) \) for \( P \in E_n^0(Q) \) satisfies

\[
\mu\left( \bigcup_{b \in \mathcal{P}_{n_0}^{(i)}(Q)} \sigma(P) \right) \leq \sum_{b \in \mathcal{P}_{n_0}^{(i)}(Q)} \sum_{P \in \mathcal{E}_n^0(Q)} \mu(\sigma(P)) \leq \sum_{b \in \mathcal{P}_{n_0}^{(i)}(Q)} \sum_{P \in \mathcal{E}_n^0(Q)} \frac{nc_0^3 Q^{-2v_0 + v_1 + u_1 + u_2}}{c_1} \mu(\sigma_1(P))
\]

\[
\leq \frac{3n-1}{c_0 c_1} Q^{n-1} Q^{-2v_0 + v_1 + u_2} \mu(I \times K) = \frac{3n-1}{c_0 c_1} \mu(I \times K).
\]

Thus, by choosing \( c_1 = \frac{4p^4 c_0^3 n}{1-k} \), the measure of the set of points lying in sets \( \sigma(P) \) for \( P \in \bigcup_{b} E_n^0(Q) \) is at most \( \frac{3n-1}{c_0 c_1} \mu(I \times K) \).

Now, let \( P \in \mathcal{P}_n^{(i)}(Q) \). Then there exists \( P' \in \mathcal{P}_n^{(i)}(Q) \) such that \( \mu(\sigma_1(P) \cap \sigma_1(P')) \geq \frac{1}{2} \mu(\sigma_1(P)) \). Let \( R = P - P' \) so that \( R(f) = b_1 f + b_0 \). Then, \( R \) satisfies

\[
|b_1 x + b_0| \leq 4c_1 Q^{-u_1}, \quad |R'(x)| = |b_1| \leq 6c_0 Q^{1-v_1},
\]

\[
|b_1 w + b_0| < Q^{-u_2}, \quad |R'(w)| = |b_1| < 3\delta_0 Q^{-v_1}
\]

on \( \sigma_1(P) \cap \sigma_1(P') \). From this it follows that \( |b_1| \leq 6c_0 Q^{1-v_1} \). Define \( s_1 \) and \( s_2 \) such that \( p^{s_1} < Q < p^{s_1 + 1} \) and \( p^{s_2} < \delta_0 < p^{s_2 + 1} \). Also note that \( 1 \leq 3 \leq p^2 \) for all primes \( p \). Let \([\cdot] \) denote the integer part. Then, as \( |b_1| < 3\delta_0 Q^{-v_1} < p^{s_2 + 3 - [s_1 v_1]} \) we have \( b_1 = p^{s_2} b'_1 \) for some integer \( b'_1 \) with \( \{b'_1, p\} = 1 \) and \( L \geq \{s_1 v_1\} - s_2 - 3 \). Since \( K \) is a cylinder we can write \( K = B(c, p^{-i}) \) where \( c \in Z \) and \( |c|_p = p^{-T} \) for some \( T < i \). Thus, if \( w \in K \) then

\[
|w|_p = p^{-T} \quad \text{and} \quad |b_1 w|_p = p^{-T - L},
\]

There are now two cases to consider. First assume that \( p^{-T - L} > Q^{-u_2} \). Then, as \( |b_1 w + b_0|_p \leq |Q^{-u_2} \) we have \( |b_0|_p = |b_1 w|_p = p^{-T - L} \). There are now two cases to consider. First assume that \( p^{-T - L} > Q^{-u_2} \). Then, as \( |b_1 w + b_0|_p \leq |Q^{-u_2} \) we have \( |b_0|_p = |b_1 w|_p = p^{-T - L} \).

From (4.4) and previously it follows that

\[
|b'_1 x + b'_0| \leq 4c_1 p^{-L} Q^{-u_1}, \quad |b'_1 w + b'_0|_p \leq L^2 Q^{-u_2}.
\]

For an inessential polynomial \( P \) these inequalities will hold for some \( b'_1, b'_0 \). Thus, the problem has now been reduced to considering the measure of the set of points \( (x, w) \) for which the above inequalities hold for some suitable \( b'_1, b'_0 \). The measure of the set of \( (x, w) \) satisfying this system for a fixed \( b'_0 \) and \( b'_1 \) is

\[
\leq 8c_1 \frac{Q^{-u_1 - u_2}}{|b'_1| |b'_0|} \leq \frac{8c_1 Q^{-u_1 - u_2}}{|b'_1|}
\]

as \( b'_1 \) is an integer and \( \{b'_1, p\} = 1 \). Next, for a fixed \( b'_1 \), we obtain an upper bound for the number of \( b'_0 \) such that \( b'_0/b'_1 \in I \) and \( b'_0/b'_1 \in K \). From these two inclusions we have that \( b'_0 \in b'_1 I \) and \( b'_0/b'_1 = c + \sum_{i=0}^{\infty} a_i p^{i+1} \) with \( a_i \in \{0, \ldots, p - 1\} \). Assume that \( b'_0/b'_1 \) lies in both \( I \) and \( K \) and assume that \( t/b'_1 \) also lies in \( K \). Then

\[
\frac{t}{b'_1} = \frac{b'_0}{b'_1} + \sum_{i=0}^{\infty} m_i p^{i+1}
\]

with \( m_i \in \{0, \ldots, p - 1\} \). Thus \( t = b'_0 + m_0 b'_1 p + \cdots + b'_0 + b'_1 p^i \). Hence \( t - b'_0 > p^i \) and the number of \( t \) for which \( t/b'_1 \) lies in both \( I \) and \( K \) is at most \( \frac{\mu(b'_1 I)}{p} = |b'_1| \mu(I \times K) \). Therefore, summing over all \( b'_1 \) with \( |b'_1| \leq 6c_0 p^{-L} Q^{-v_1} \) we have that the set of \( (x, w) \) satisfying this
system has measure at most

\[ 48c_0 c_1 p^{-u_1 - u_2 - v_1} \mu(I \times K) \leq 48c_0 c_1 \delta_p^{d+u_1} Q^{1-u_1 - u_2 - 2v_1} \mu(I \times K) \]

\[ \leq 48c_0 c_1 \delta_p^{d+u_1} \mu(I \times K) \]

from (4.3) and the definitions of \( s_1, s_2 \) and \( L \). Clearly, there exists \( \delta_0 \) such that the measure of the set of points \((x, w)\) which lie in \( \sigma_1(P) \) for at least one \( P \in I_n^c(Q) \) is at most \( \frac{1}{\delta} \mu(I \times K) \).

Now we consider the second case when \( p^{-(T+L)} \leq Q^{-u_2} \). In this case we have that \( |b_0|_p \leq Q^{-u_2} \leq p^{[s_1 u_2]} \). Hence, for \( Q \) sufficiently large, there exists \( b'_0 \in \mathbb{Z} \) such that \( b_0 = p^{[s_1 u_2]} T b'_0 \).

We can also write \( b_1 = p^{b_1'} = p^{[s_1 u_2]} T p^{[-s_1 u_2]} T b'_1 \). Let \( b'_1 = p^{L-[s_1 u_2]} T b'_1 \) so that \( \# b'_1 = \# b'_1' \leq 12c_0 p^{-L Q^{1-v_1}} \) and \( |b'_1'|_p = p^{-L-[s_1 u_2]+T} \) as \( (b'_1, p) = 1 \). Thus

\[ |b'_1 x + b'_0| \leq 4c_1 p^{-[s_1 u_2]} T Q^{-u_1}, \quad |b'_1' x + b'_0'| \leq p^{[s_1 u_2]} T Q^{-u_2}. \]

Again the measure of the set of \((x, w)\) satisfying this system for a fixed \( b'_0 \) and \( b'_1 \) is

\[ \leq 8c_1 \frac{Q^{-u_1 - u_2}}{|b'_1||b'_1'|_p} \leq 8c_1 \frac{Q^{-u_1 - u_2} p^{L-[s_1 u_2]+T}}{|b'_1|}. \]

As before the number of \( b'_0 \) for a fixed \( b'_1 \) is \( |b'_1'| \mu(I \times K) \). Finally, therefore, the measure of the set of \((x, w)\) satisfying the system is at most

\[ 96c_0 c_1 p^{-u_1 - u_2 - v_1} \mu(I \times K) = 96c_0 c_1 Q^{1-u_1 - u_2 - v_1} p^{-[s_1 u_2]+T} \mu(I \times K). \]

Using the definition of \( s_1 \) this is

\[ \leq 96c_0 c_1 p^{u_2 + 1 + T} Q^{1-u_1 - 2u_2 - v_1} \mu(I \times K), \]

which can be made arbitrarily small for \( Q \) sufficiently large by (4.3). This completes the proof of the theorem.

References
