A note on operator tuples which are $(m, p)$-isometric as well as $(\mu, \infty)$-isometric

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Abstract

We show that if a tuple of commuting, bounded linear operators $(T_1, \ldots, T_d) \in B(X)^d$ is both an $(m, p)$-isometry and a $(\mu, \infty)$-isometry, then the tuple $(T_1^m, \ldots, T_d^m)$ is a $(1, p)$-isometry. We further prove some additional properties of the operators $T_1, \ldots, T_d$ and show a stronger result in the case of a commuting pair $(T_1, T_2)$.

Keywords: operator tuple, normed space, Banach space, $m$-isometry, $(m, p)$-isometry, $(m, \infty)$-isometry

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1 Introduction

Let in the following $X$ be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let the symbol $\mathbb{N}$ denote the natural numbers including 0.

A tuple of commuting linear operators $T := (T_1, \ldots, T_d)$ with $T_j : X \to X$ is called an $(m, p)$-isometry (or an $(m, p)$-isometric tuple) if, and only if, for given $m \in \mathbb{N}$ and $p \in (0, \infty)$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_d!} \|T^\alpha x\|^p = 0, \quad \forall x \in X. \quad (1.1)$$

Here, $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multi-index, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ the sum of its entries, $k! := \frac{k!}{\alpha_1! \cdots \alpha_d!}$ a multinomial coefficient and $T^\alpha := T_1^{\alpha_1} \cdots T_d^{\alpha_d}$, where $T_j^0 := I$ is the identity operator.

Tuples of this kind have been introduced by Gleason and Richter [10] on Hilbert spaces (for $p = 2$) and have been further studied on general normed spaces in [8]. The tuple case generalises the single operator case, originating in the works of Richter [11] and Agler [2] in the 1980s and being comprehensively studied in the Hilbert space case by Agler and Stankes [3]; the single operator case on Banach spaces has been introduced by Bayart in [4] in its general form and also has also been studied in [7] and [12]. We remark that boundedness, although usually assumed, is not essential for the definition of $(m, p)$-isometries, as shown by Bermúdez, Martínón and Müller in [5]. Boundedness does, however, play an important role in the theory of objects of the following kind:

Let $B(X)$ denote the algebra of bounded (i.e. continuous) linear operators on $X$. Equating sums over even and odd $k$ and then considering $p \to \infty$ in
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(1.1), leads to the definition of \((m, \infty)\)-isometries (or \((m, \infty)\)-isometric tuples). That is, a tuple of commuting, bounded linear operators \(T \in B(X)^d\) is referred to as an \((m, \infty)\)-isometry if, and only if, for given \(m \in \mathbb{N}\) with \(m \geq 1\),

\[
\max_{|\alpha| = 0, \ldots, m} \|T^\alpha x\| = \max_{|\alpha| = 0, \ldots, m} \|T^\alpha x\|, \quad \forall x \in X. \tag{1.2}
\]

These tuples have been introduced in [8], with the definition of the single operator case appearing in [9]. Although, it may be possible that tuples of unbounded operators satisfying (1.2) exist, several important statements on \((m, \infty)\)-isometries require boundedness. Therefore, from now on, we will always assume the operators \(T_1, \ldots, T_d\) to be bounded.

In [8], the question is asked what necessary properties a commuting tuple \(T \in B(X)^d\) has to satisfy if it is both an \((m, p)\)-isometry and a \((\mu, \infty)\)-isometry, where possibly \(m \neq \mu\). In the single operator case this question is trivial and answered in [9]: If \(T = T_1\) is a single operator, then the condition that \(T_1\) is an \((m, p)\)-isometry is equivalent to the mapping \(n \mapsto \|T_1^n x\|^p\) being a polynomial of degree \(\leq m - 1\) for all \(x \in X\). This has been already been observed for operators on Hilbert spaces in [10] and shown in the Banach space/normed space case in [9]: the necessity of the mapping \(n \mapsto \|T_1^n x\|^p\) being a polynomial has already been proven in [11] and [6]. On the other hand, in [9] it is shown that if a bounded operator \(T = T_1 \in B(X)\) is a \((\mu, \infty)\)-isometry, then the mapping \(n \mapsto \|T_1^n x\|^\mu\) is bounded for all \(x \in X\). The conclusion is obvious: if \(T = T_1 \in B(X)\) is both \((m, p)\)- and \((\mu, \infty)\)-isometric, then \(n \mapsto \|T_1^n x\|^p\) is always constant and \(T_1\) has to be an isometry (and, since every isometry is \((m, p)\)- and \((\mu, \infty)\)-isometric, we have equivalence).

The situation is, however, far more difficult in the multivariate, that is, in the operator tuple case. Again, we have equivalence between \(T = (T_1, \ldots, T_d)\) being an \((m, p)\)-isometry and the mapping \(n \mapsto \sum_{|\alpha| = n} \frac{n!}{\alpha!} \|T^\alpha x\|^p\) being polynomial of degree \(\leq m - 1\) for all \(x \in X\). The necessity part of this statement has been proven in the Hilbert space case in [10] and equivalence in the general case has been shown in [8]. On the other hand, one can show that if \(T \in B(X)^d\) is a \((\mu, \infty)\)-isometry, then the family \((\|T^n x\|)_{n \in \mathbb{N}}\) is bounded for all \(x \in X\), which has been proven in [8]. But this fact only implies that the polynomial growth of \(n \mapsto \sum_{|\alpha| = n} \frac{n!}{\alpha!} \|T^\alpha x\|^p\) has to caused by the factors \(\frac{n!}{\alpha!}\) and does not immediately give us any further information about the tuple \(T\).

There are several results in special cases proved in [8]. For instance, if a commuting tuple \(T = (T_1, \ldots, T_d) \in B(X)^d\) is an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry and we have \(m = 1\) or \(\mu = 1\) or \(m = \mu = d = 2\), then there exists one operator \(T_{j_0} \in \{T_1, \ldots, T_d\}\) which is an isometry and the remaining operators \(T_k\) for \(k \neq j_0\) are in particular nilpotent of order \(m\). Although, we are not able to obtain such a results for general \(m \in \mathbb{N}\) and \(\mu, d \in \mathbb{N} \setminus \{0\}\), yet, we can prove a weaker property: In all proofs of the cases discussed in [8], the fact that the tuple \((T_1^m, \ldots, T_d^m)\) is a \((1, p)\)-isometry is of critical importance (see the proofs of [8] Theorem 7.1 and Proposition 7.3]). We will show in this paper that this fact holds in general for any tuple which is both \((m, p)\)-isometric and \((\mu, \infty)\)-isometric, for general \(m, \mu\) and \(d\).

The notation we will be using is basically standard, with one possible exception: We will denote the tuple of \(d - 1\) operators obtained by removing one operator \(T_{j_0}\) from \((T_1, \ldots, T_d)\) by \(T'_{j_0}\), that is \(T'_{j_0} := (T_1, \ldots, T_{j_0-1}, T_{j_0+1}, \ldots, T_d) \in B(X)^{d-1}\).
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\[ B(X)^{d-1} \] (not to be confused with the dual of the operator \(T_{j_0}\), which will not appear in this paper). Analogously, we denote by \(\alpha'_{j_0}\) the multi-index obtained by removing \(\alpha_{j_0}\) from \((\alpha_1, \ldots, \alpha_d)\).

We will further use the notations \(R(T_j)\) for the range and \(N(T_j)\) for the kernel (or nullspace) of an operator \(T_j\).

## 2 Preliminaries

In this section, we introduce two needed definitions/notations and compile a number of propositions and theorems, predominantly taken from [8], which are necessary for our considerations.

In the following, for \(T \in B(X)^d\) and given \(p \in (0, \infty)\), define for all \(x \in X\) the sequences \((Q^{n,p}(T, x))_{n \in \mathbb{N}}\) by

\[
Q^{n,p}(T, x) := \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p.
\]

Define further for all \(\ell \in \mathbb{N}\) and all \(x \in X\), the mappings \(P^{(p)}_\ell(T, \cdot) : X \to \mathbb{R}\), by

\[
P^{(p)}_\ell(T, x) := \sum_{k=0}^\ell (-1)^{\ell-k} \binom{\ell}{k} Q^{k,p}(T, x) = \sum_{k=0}^\ell (-1)^{\ell-k} \binom{\ell}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p.
\]

It is clear that \(T \in B(X)^d\) is an \((m, p)\)-isometry if, and only if, \(P^{(p)}_m(T, \cdot) \equiv 0\).

If the context is clear, we will simply write \(P_\ell(x)\) and \(Q^n(x)\) instead of \(P^{(p)}_\ell(T, x)\) and \(Q^{n,p}(T, x)\).

Further, for \(n, k \in \mathbb{N}\), define the (descending) Pochhammer symbol \(n^{(k)}\) as follows:

\[
n^{(k)} := \begin{cases} 0, & \text{if } k > n, \\ \left(\begin{array}{c} n \\ k \end{array}\right) \frac{k!}{\alpha!}, & \text{else}. \end{cases}
\]

Then \(n^{(0)} = 0^{(0)} = 1\) and, if \(n, k > 0\) and \(k \leq n\), we have

\[
n^{(k)} = n(n-1) \cdots (n-k+1).
\]

As mentioned above, a fundamental property of \((m, p)\)-isometries is that their defining property can be expressed in terms of polynomial sequences.

**Theorem 2.1** ([8, Theorem 3.1]). \(T \in B(X)^d\) is an \((m, p)\)-isometry if, and only if, there exists a family of polynomials \(f_x : \mathbb{R} \to \mathbb{R}\), \(x \in X\), of degree \(\leq m - 1\) with \(f_x|_N = (Q^n(x))_{n \in \mathbb{N}}\).

This actually follows by the [not immediate](#) application of a well-known theorem about functions defined on the natural numbers, which itself will be needed for our considerations as well. We give it here in a simplified form which is sufficient for our needs.

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1. Set \(\text{deg} 0 := -\infty\) to account for the case \(m = 0\).
2. The application of Theorem 2.2 to \((m, p)\)-isometries by setting \(a = (Q_n(x))_{n \in \mathbb{N}}\) is not immediate, since the requirement \(P^{(p)}_n(T, x) = 0\) is only the case \(n = 0\) in [2.2].
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**Theorem 2.2** (see, for instance, [1 Satz 3.1]). Let \(a = (a_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) be a sequence and \(m \in \mathbb{N}\). Then we have

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} = 0, \quad \forall n \in \mathbb{N}
\]

(2.1)

if, and only if, there exists a polynomial function \(f\) of degree \(\deg f \leq m - 1\) with \(f|_n = a_n\).

Two important consequences of Theorem 2.1 are contained in the following corollary. The first part describes the Newton-form of the Lagrange-polynomial \(f_x\) interpolating \((Q^a(x))_{n \in \mathbb{N}}\). The second part trivially describes the leading coefficient of \(f_x\).

**Corollary 2.3** ([8 Proposition 3.2]). Let \(m \geq 1\) and \(T \in B(X)^d\) be an \((m, p)\)-isometry. Then we have

(i) for all \(n \in \mathbb{N}\)

\[
Q^n(x) = \sum_{k=0}^{m-1} n^{(k)} \left( \frac{1}{k!} P_k(x) \right), \quad \forall x \in X;
\]

(ii)

\[
\lim_{n \to \infty} \frac{Q^n(x)}{n^{m-1}} = \frac{1}{(m-1)!} P_{m-1}(x) \geq 0, \quad \forall x \in X.
\]

Regarding \((m, \infty)\)-isometries, we will need the following two statements.

**Theorem 2.5** ([8, Proposition 5.5, Theorem 5.1 and Remark 5.2]). Let \(T = (T_1, \ldots, T_d) \in B(X)^d\) be an \((m, \infty)\)-isometric tuple. Define the norm \(\|\cdot\|_{\infty}: X \to [0, \infty)\) via \(\|x\|_{\infty} := \max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\|\), for all \(x \in X\), and denote

\[
X_{j,|\cdot|_{\infty}} := \{ x \in X \mid \|x\|_{\infty} = \|T_j^n x\|_{\infty} \text{ for all } n \in \mathbb{N} \}.
\]

Then

\[
X = \bigcup_{j=1, \ldots, d} X_{j,|\cdot|_{\infty}}.
\]

(Note that, by Proposition 2.3, \(\|\cdot\|_{\infty} = \|\cdot\|\) if \(m = 1\).)

We will also require a fundamental fact on tuples which are both \((m, p)\)- and \((\mu, \infty)\)-isometric and an (almost) immediate corollary.

**Lemma 2.6** ([8 Lemma 7.2]). Let \(T = (T_1, \ldots, T_d) \in B(X)^d\) be an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry. Let \(\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d\) be a multi-index with the property that \(|\gamma_j| \geq m\) for every \(j \in \{1, \ldots, d\}\). Then \(T^\gamma = 0\).
Conversely, this implies that if an operator $T^\alpha$ is not the zero-operator, the multi-index $\alpha$ has to be of a specific form. The proof in [8] of the following corollary appears to be overly complicated, the statement is just the negation of the previous lemma.

**Corollary 2.7 ([8, Corollary 7.1]).** Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m, p)$-isometry for some $m \geq 1$ as well as a $(\mu, \infty)$-isometry. If $\alpha \in \mathbb{N}^d$ is a multi-index with $T^\alpha \neq 0$ and $|\alpha| = n$, then there exists some $j_0 \in \{1, ..., d\}$ with $T^\alpha = T_{j_0}^{n - |\alpha'|} (T_{j_0})^{\alpha'_0}$ and $|\alpha'_0| \leq m - 1$.

This fact has consequences for the appearance of elements of the sequences $(Q^n(x))_{n \in \mathbb{N}}$, since several summands become zero for large enough $n$. That is, we have trivially by definition of $(Q^n(x))_{n \in \mathbb{N}}$:

**Corollary 2.8 ([8, proof of Theorem 7.1]).** Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m, p)$-isometry for some $m \geq 1$ as well as a $(\mu, \infty)$-isometry. Then, for all $n \in \mathbb{N}$ with $n \geq 2m - 1$, we have

$$Q^n(x) = \sum_{\beta \in \mathbb{N}^d} \sum_{|\beta| = 0, ..., m - 1} \frac{n!}{(n - |\beta|)! |\beta|!} ||T_j^{n - |\beta|} (T_j)^\beta x||^p, \forall x \in X,$$

where $\frac{n!}{(n - |\beta|)! |\beta|!} = \binom{n - |\beta|}{|\beta|}$. (We set $n \geq 2m - 1$ to ensure that every multi-index only appears once.)

### 3 The main result

We first present the main result of this article, which is a generalisation of [8, Proposition 7.3], before stating a preliminary lemma needed for its proof.

**Theorem 3.1.** Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m, p)$-isometric as well a $(\mu, \infty)$-isometric tuple. Then

(i) the sequences $n \mapsto ||T_j^n x||$ become constant for $n \geq m$, for all $j \in \{1, ..., d\}$, for all $x \in X$.

(ii) the tuple $(T_1^m, ..., T_d^m)$ is a $(1, p)$-isometry, that is

$$\sum_{j=1}^d ||T_j^m x||^p = ||x||^p, \forall x \in X.$$

(iii) for any $(n_1, ..., n_d) \in \mathbb{N}^d$ with $n_j \geq m$ for all $j$, the operators $\sum_{j=1}^d T_j^{n_j}$ are isometries, that is

$$\left\| \sum_{j=1}^d T_j^{n_j} x \right\| = ||x||, \forall x \in X.$$
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Of course, (i) and (ii) imply that, for any \((n_1, \ldots, n_d) \in \mathbb{N}^d\) with \(n_j \geq m\) for all \(j\),

\[
\sum_{j=1}^{d} \|T_{n_j}^j x\|^p = \|x\|^p, \quad \forall x \in X,
\]

Theorem 3.1 is a consequence of the following lemma, which is a weaker version of 3.1(i).

**Lemma 3.2.** Let \(T = (T_1, \ldots, T_d) \in B(X)^d\) be an \((m, p)\)-isometric as well as a \((\mu, \infty)\)-isometric tuple. Let further \(\kappa \in \mathbb{N}^{d-1}\) be a multi-index with \(|\kappa| \geq 1\).

Then the mappings

\[
n \mapsto \|T_n^n (T_j)^{\kappa} x\|
\]

become constant for \(n \geq m\), for all \(j \in \{1, \ldots, d\}\), for all \(x \in X\).

**Proof.** If \(m = 0\), then \(X = \{0\}\) and if \(m = 1\), the statement holds trivially, since \(T_j T_i = 0\) for all \(i \neq j\) by Lemma 2.6. So assume \(m \geq 2\). Further, it clearly suffices to consider \(|\kappa| = 1\), since the statement then holds for all \(x \in X\). The proof, however, works by proving the theorem for \(|\kappa| \in \{1, \ldots, m-1\}\) in descending order. (Note that the case \(|\kappa| \geq m\) is also trivial, again by Lemma 2.6.)

Since for \(n \geq 2m - 1\), by Corollary 2.8,

\[
Q^n(x) = \sum_{\beta \in \mathbb{N}^{d-1}} \sum_{j=1}^{d} \frac{n(\beta)}{\beta!} \|T_j^n (T_j)^{\beta} x\|^p, \quad \forall x \in X,
\]

and \(P_{m-1}(x) = \lim_{n \to \infty} \frac{Q^n(x)}{n}\), for all \(x \in X\), by Corollary 2.3(ii), we have that

\[
P_{m-1}(x) = \lim_{n \to \infty} \sum_{\beta \in \mathbb{N}^{d-1}} \sum_{j=1}^{d} \frac{1}{\beta!} \|T_j^n (T_j)^{\beta} x\|^p, \quad \forall x \in X.
\]

Now fix an arbitrary \(j_0 \in \{1, \ldots, d\}\) and let \(\kappa \in \mathbb{N}^{d-1}\) with \(|\kappa| \in \{1, \ldots, m-1\}\). Again, by Lemma 2.6 we have, for any \(\nu \geq 1\),

\[
P_{m-1} (T_{j_0}^{\nu} (T_{j_0})^\kappa x) = 0, \quad \forall x \in X.
\]

Now let \(\nu \geq m\) and set \(\ell := m - |\kappa|\). Then \(\ell \in \{1, \ldots, m-1\}\) and \(|\kappa| = m - \ell\).
We again apply Lemma 2.6 this time to $Q^k(T^\nu_{j_0} (T'_{j_0})^\kappa x)$. By definition,

$$Q^k(T^\nu_{j_0} (T'_{j_0})^\kappa x) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \| T^\alpha (T^\nu_{j_0} (T'_{j_0})^\kappa x) \|^p$$

$$= \| T^k_{j_0} (T^\nu_{j_0} (T'_{j_0})^\kappa x) \|^p + \sum_{j=1}^k \sum_{\beta \in \mathbb{N}^{d-1}} \frac{k!}{(k-j)!|\beta|!} \| T^{k-j}_{j_0} (T^\nu_{j_0} (T'_{j_0})^\kappa x) \|^p$$

$$\geq \| T^{\nu+k}_{j_0} (T'_{j_0})^\kappa x \|^p + \sum_{j=1}^\ell \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{\beta!} \| T^{\nu+k-j}_{j_0} (T'_{j_0})^\kappa x \|^p$$

for all $k \in \mathbb{N}$, for all $x \in X$. Here, in the third line, the fact that $\nu \geq m$ is used, where in the last line, we utilise the fact that $k^{(j)} = 0$ if $j > k$.

We now prove our statement by (finite) induction on $\ell$.

\(\ell = 1\):

For $\ell = 1$ and $|\kappa| = m - 1$, we have

$$Q^k (T^\nu_{j_0} (T'_{j_0})^\kappa x) = \| T^{\nu+k}_{j_0} (T'_{j_0})^\kappa x \|^p, \quad \forall k \in \mathbb{N}, \forall x \in X.$$ 

Hence, since $P_{m-1}(x) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} Q^k(x)$ by definition, we have, by \(\text{[2.1]}\),

$$P_{m-1} (T^\nu_{j_0} (T'_{j_0})^\kappa x) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \| T^{\nu+k}_{j_0} (T'_{j_0})^\kappa x \|^p = 0, \quad \forall x \in X.$$ 

However, by definition, that means, that the operator $T_{j_0}|_{R(T^\nu_{j_0} (T'_{j_0})^\kappa)}$ (that is, $T_{j_0}$ restricted to the range of $T^\nu_{j_0} (T'_{j_0})^\kappa x$) is an $(m-1, p)$-isometric operator.

By Theorem 2.1 (or, as mentioned in the introduction, by statements proven by earlier authors), this implies that the sequences $n \mapsto \| T^{n+\nu}_{j_0} (T'_{j_0})^\kappa x \|^p$ is polynomial of degree $\leq m - 2$, for all $x \in X$. Thus, $n \mapsto \| T^n_{j_0} (T'_{j_0})^\kappa x \|^p$, become polynomial of degree $\leq m - 2$, for $n \geq \nu \geq m$, for all $x \in X$.

However, since $T$ is a $(\mu, \infty)$-isometric tuple, by Proposition 2.4 the sequences $n \mapsto \| T^n x \|$ are bounded for all $j \in \{1, ..., d\}$, for all $x \in X$. Therefore, we must have that the mappings

$$n \mapsto \| T^n_{j_0} (T'_{j_0})^\kappa x \|$$

become constant for $n \geq m$, for all $x \in X$.

Since $\ell \in \{1, ..., m-1\}$, if we had $m = 2$, we are already done. So assume in the following that $m \geq 3$.

\(\ell \to \ell + 1\):
Assume that the statement holds for some \( \ell \in \{1, ..., m - 2\} \). That is, for all \( \kappa \in \mathbb{N}^{d-1} \) with \( |\kappa| = m - \ell \) the sequences

\[
  n \mapsto ||T^n_{j_0} (T'_{j_0})^\kappa x||
\]

become constant for \( n \geq m \), for all \( x \in X \).

Now take a multi-index \( \tilde{\kappa} \in \mathbb{N}^{d-1} \) with \( |\tilde{\kappa}| = m - (\ell + 1) \) and consider

\[
  Q^k(T^\nu_{j_0} (T'_{j_0})^{\tilde{\kappa}} x) = ||T^{\nu+k}_{j_0} (T'_{j_0})^{\tilde{\kappa}} x||^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{|\beta|} ||T^{\nu+k-j}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p.
\]

Note that we have \( |\tilde{\kappa} + \beta| \geq m - \ell \), since \( |\beta| \geq 1 \). Hence, if \( k \geq j \), by our induction assumption,

\[
  ||T^{\nu+k-j}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p = ||T^\nu_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p, \ \forall x \in X,
\]

since \( n \mapsto ||T^n_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x|| \) become constant for \( n \geq \nu \geq m \).

Hence, we have

\[
  Q^k(T^\nu_{j_0} (T'_{j_0})^{\tilde{\kappa}} x) = ||T^{\nu+k}_{j_0} (T'_{j_0})^{\tilde{\kappa}} x||^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{|\beta|} ||T^{\nu}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p.
\]

Then, by definition and 3.1,

\[
  0 = P_{m-1} \left( T^\nu_{j_0} (T'_{j_0})^{\tilde{\kappa}} x \right) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} Q^k(x)
  = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} ||T^{\nu+k}_{j_0} (T'_{j_0})^{\tilde{\kappa}} x||^p
  + \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left( \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{|\beta|} ||T^{\nu}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p \right),
\]

for all \( x \in X \). But now, for all \( x \in X \), the sequence

\[
  k \mapsto \left( \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{|\beta|} ||T^{\nu}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p \right)
\]

is polynomial (in \( k \)) of degree \( \leq \ell - 1 \leq m - 3 \) (with trailing coefficient 0).

Hence, by Theorem 2.2,

\[
  \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left( \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\beta \in \mathbb{N}^{d-1}} \frac{1}{|\beta|} ||T^{\nu}_{j_0} (T'_{j_0})^{\beta+\tilde{\kappa}} x||^p \right) = 0
\]
and, thus,

$$0 = P_{m-1} \left( T_{j_0}^\nu (T_{j_0}')^{\bar{k}} x \right) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left\| T_{j_0}^{-k} (T_{j_0}')^{\bar{k}} x \right\|^p,$$

for all $x \in X$. Now we can repeat the argument from the case $\ell = 1$ (that is, $T_{j_0}$ restricted to the range of $T_{j_0}^\nu (T_{j_0}')^{\bar{k}}$ is an $(m-1,p)$-isometric operator), to obtain again that the sequences

$$n \mapsto \left\| T_{j_0}^n (T_{j_0}')^{\bar{k}} x \right\|$$

become constant for $n \geq \nu \geq m$, for all $x \in X$. This concludes the induction step and the proof. \qed

We can now prove the main result.

Proof of Theorem 3.1. By the lemma above, we have for $n \geq 2m-1$,

$$Q^n(x) = \sum_{\beta \in \mathbb{N}^{d-1}} n^{(\beta)} \sum_{|\beta|=0, \ldots, m-1} \frac{1}{\beta!} \left\| T_j^{-|\beta|} (T_j')^\beta x \right\|^p$$

$$= \sum_{\beta \in \mathbb{N}^{d-1}} n^{(\beta)} \sum_{|\beta|=1, \ldots, m-1} \frac{1}{\beta!} \left\| T_j^m (T_j')^\beta x \right\|^p + \sum_{j=1}^d \left\| T_j^nx \right\|^p, \quad \forall x \in X. \quad (3.2)$$

That is, for all $x \in X$, for $n \geq m-1$, the sequences $n \mapsto Q^n(x)$ are almost polynomial (of degree $\leq m-1$), with the term $\sum_{j=1}^d \left\| T_j^nx \right\|^p$ instead of a (constant) trailing coefficient.

However, by Corollary 2.3.(i), we know that for any $x \in X$, the sequence $n \mapsto Q^n(x)$ are indeed polynomial. Since, by Proposition 2.4, for each $x \in X$, the sequence $n \mapsto \sum_{j=1}^d \left\| T_j^nx \right\|^p$ is bounded, we can successive compare and remove coefficients of the formula for $Q_n(x)$ as given in 2.3.(i) and (3.2), until we eventually obtain that

$$\sum_{j=1}^d \left\| T_j^nx \right\|^p = \left\| x \right\|^p, \quad \forall x \in X, \forall n \geq 2m-1. \quad (3.3)$$

Since $T_i^m T_j^m = 0$ for all $i \neq j$, by Lemma 2.6 replacing $x$ by $T_j^\nu x$ with $\nu \geq m$ in this last equation, gives $\left\| T_j^\nu x \right\| = \left\| T_j^{n+\nu} x \right\|$ for all $n \geq 2m-1$, for all $x \in X$.

Hence, the sequences $n \mapsto \left\| T_j^nx \right\|$ become constant for $n \geq m$, for all $j \in \{1, \ldots, d\}$, for all $x \in X$. This is 3.1.(i).

But then, (3.3) becomes

$$\sum_{j=1}^d \left\| T_j^nx \right\|^p = \left\| x \right\|^p, \quad \forall x \in X. \quad \text{This is 3.1(ii).}$$
A note on operator tuples which are \((m, p)\)-isometric and \((\mu, \infty)\)-isometric

Now take any \((n_1, \ldots, n_d) \in \mathbb{N}^d\) with \(n_j \geq m\) for all \(j\) and replace \(x\) in the equation above by \(\sum_{j=1}^d T_j^{n_j}\). Then, again, since \(T_i^n T_j^m = 0\) for \(i \neq j\), and since \(n \mapsto \|T_j^n x\|\) become constant for \(n \geq m\),

\[
\sum_{j=1}^d \|T_j^{n_j} x\|^p = \sum_{j=1}^d \|T_j^n x\|^p = \| \sum_{j=1}^d T_j^{n_j} x\|^p, \quad \forall x \in X.
\]

Together with 3.1(i), this implies 3.1(iii). \(\square\)

It is clear that we have a stronger result if one of the operators \(T_{j_0} \in \{T_1, \ldots, T_d\}\) is surjective. Theorem 3.1(i) then forces this operator to be an isometric isomorphism and by 3.1(ii) the remaining operators are nilpotent.

If one of the operators \(T_{j_0} \in \{T_1, \ldots, T_d\}\) is injective, by Lemma 2.6 and 3.1(ii) we obtain at least that \(T_{j_0}^m\) is an isometry and the remaining operators are nilpotent. However, while, by definition of an \((m, p)\)-isometry, we must have \(\bigcap_{j=1}^d \text{N}(T_j) = \{0\}\), it is not clear that the kernel of a single operator has to be trivial.

4 Some further remarks and the case \(d = 2\)

We finish this note with a stronger result for the case of a commuting pair \((T_1, T_2) \in B(X)^d\). We first state the following two easy corollaries of Theorem 3.1 which hold for general \(d\).

Corollary 4.1. Let \(T = (T_1, \ldots, T_d) \in B(X)^d\) be an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry. Then \(T_j^m = 0\) or \(\|T_j^m\| = 1\) for any \(j \in \{1, \ldots, d\}\).

Proof. By Theorem 3.1(ii) we have \(\|T_j^m\| \leq 1\) for any \(j\). On the other hand, by 3.1(i) we have

\[
\|T_j^m x\| = \|T_j^{m+1} x\| \leq \|T_j^m\| \cdot \|T_j^m x\|, \quad \forall x \in X,
\]

for any \(j\). That is, \(T_j^m = 0\) or \(\|T_j^m\| \geq 1\). \(\square\)

Lemma 4.2. Let \(T = (T_1, \ldots, T_d) \in B(X)^d\) be an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry. Define \(\|\cdot\| : X \to [0, \infty)\) and \(X_{\mu, \infty}\) as in Theorem 2.6. Then

\[
X_{\mu, \infty} = \{ x \in X \mid \exists \alpha(x) \in \mathbb{N}^d, \ s.t. \ |\alpha(x)| \leq \mu - 1 \ \text{and} \ \|x\|_{\infty} = \|T_j^m (T_j^\alpha x)\|, \ \forall \alpha \in \mathbb{N} \}.
\]

Proof. By Proposition 2.4 we know that for every \(x \in X\), there exists an \(\alpha(x) \in \mathbb{N}^d\) with \(\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|T^{\alpha(x)} x\|\) and \(|\alpha(x)| \leq \mu - 1\).

Then \(x \in X_{\mu, \infty}\) if, and only if, for all \(n \in \mathbb{N}\), there exists an \(\alpha(x, n) \in \mathbb{N}^d\) with \(|\alpha(x, n)| \leq \mu - 1\) s.t. \(\|x\|_{\infty} = \|T_j^m T^{\alpha(x, n)} x\|\). Hence, the inclusion \(\supseteq\) is clear.

To show \(\subseteq\) let \(\theta \neq x \in X_{\mu, \infty}\). Then \(T_j^m \neq 0\) and, hence, \(\|T_j^m\| = 1\).

Since \(|\alpha(x, n)| \leq \mu - 1\) for all \(n \in \mathbb{N}\), there are only finitely many choices for each \(\alpha(x, n)\). Thus, there exists an \(\alpha(x) \in \mathbb{N}^d\) and an infinite set \(M(x) \subset \mathbb{N}\) s.t.

\[
|\alpha(x)| \leq \mu - 1 \ \text{and} \ \|x\|_{\infty} = \|T_j^m T^{\alpha(x)} x\|, \ \forall n \in M(x).
\]
By Theorem 3.1 (i), $M(x)$ contains all $n \geq m$ and further,
\[
\|T^n_j T^\alpha(x)x\| = \|T^n_j (T_j)^\alpha(x)x\|, \text{ for all } n \geq m.
\]
Since $\|T^n_j\| = 1$, the statement holds for all $n \in \mathbb{N}$.

\begin{proposition}
Let $T = (T_1, T_d) \in B(X)^d$ be both an $(m, p)$-isometric and a $(\mu, \infty)$-isometric pair. Then $T^n_1$ is an isometry and $T^n_2 = 0$ or vice versa.
\end{proposition}

\begin{proof}
By Theorem 2.5, we have $X = X_{1,1,|\infty} \cup X_{2,|\infty}$.

Let $x_1 \in X_{1,|\infty}$. Then, by the previous lemma, there exists an $\alpha_2(x_1) \in \mathbb{N}$ with $T_2(x_1) \leq \mu - 1$ s.th. $\|x_1\| = \|T^n_2 T_2^{\alpha_2(x_1)}x_1\|$ for all $n \in \mathbb{N}$.

Furthermore, we have $\|x\|^p = \|T^n_2 x\|^p + \|T_2^n x\|^p$, for all $x \in X$, by Theorem 3.1 (ii). Replacing $x$ by $T_2^{\alpha_2(x_1)}x_1$ gives
\[
\|T_2^{\alpha_2(x_1)}x_1\| = \|T^n_2 T_2^{\alpha_2(x_1)}x_1\| + \|T_2^{n+\alpha_2(x_1)}x_1\|
\Rightarrow \|T_2^{\alpha_2(x_1)}x_1\| = \|x_1\| + \|T_2^n x_1\|.
\]
This implies $\|T_2^{\alpha_2(x_1)}x_1\| = \|x_1\| \infty$ and, moreover, $\|T_2^n x_1\| = 0$.

An analogous argument shows that $X_{2,|\infty} \subset N(T^n_1)$. Hence,
\[
X = N(T^n_1) \cup N(T^n_2),
\]
which forces $T^n_1 = 0$ or $T^n_2 = 0$. The statement follows from $\|x\|^p = \|T^n_2 x\|^p + \|T^n_2 x\|^p$, for all $x \in X$.
\end{proof}

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A note on operator tuples which are \((m, p)\)-isometric and \((\mu, \infty)\)-isometric


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